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HISTORY

*Tietze, Heinrich. **Gelöste und ungelöste mathematische Probleme aus alter und neuer Zeit.** Biederstein Verlag, München, 1949. Band I, xx+256 pp. (5 plates); Band II, iv+305 pp. (4 plates).

The author reproduces "vierzehn Vorlesungen für Laien und für Freunde der Mathematik." The topics are as follows: Prime numbers (two lectures); geodesics on surfaces; trisection of an angle; "neighboring regions" and the four color problem (two lectures); squaring the circle; dimension theory; scales of notation and irrational numbers; the regular 17-gon; solution of equations by radicals; infinity in mathematics; Fermat's last theorem; the "curvature of space." The book is a model for works of its kind. The lectures are detailed, charmingly written, and entertaining and informative even for a mathematician, particularly when read together with their profuse notes, which give fuller explanations, proofs, historical, biographical and bibliographical material, or philosophical remarks. No pains seem to have been spared to make the information understandable, accurate and up-to-date. There are many diagrams in the text, and the plates range from portraits of mathematicians to photographs of a tetrahedron dissected into 6 mutually intersecting convex sets and a picture in 8 colors of the 8-color problem for a sphere with two handles. There is a detailed index. *R. P. Boas, Jr.*

*Callandreau, Édouard. **Célèbres problèmes mathématiques.** Éditions Albin Michel, Paris, 1949. 478 pp. 1500 francs.

The author has assembled what amounts to an annotated anthology of famous problems, although the solutions (when available) are presented in modern form. He attempts in this way to introduce the historical method into mathematics. The problems are arranged under the headings of Arithmetic, Algebra and analysis, Geometry, Analytic geometry, Mechanics, Celestial mechanics and astronomy, Maxima and minima. They range from puzzles to general theories, from trivial problems to difficult and unsolved ones. Proofs are given when possible. There is little pretension to originality in the discussion of the individual problems; many of the discussions are taken with little change from standard treatises. The following problems are a sample of the 111 which are discussed: Newton's problem of the cows and the pastures, Fermat's last theorem, Sturm's theorem, the Dirichlet problem, Pascal's hexagon, the four-color problem, trisection of angles, spherical trigonometry, the composition of forces, Hamilton's principle, the date of Easter, the three-body problem, continuous non-differentiable functions, the brachistochrone. The author's information is not always up-to-date, even as of 1942–43, when the book was written. *R. P. Boas, Jr.* (Evanston, Ill.).

Nevanlinna, Rolf. Leitende Gesichtspunkte in der Entwicklung der Mathematik. Vierteljschr. Naturforsch. Ges. Zürich **95**, 1–22 (1950).

Weil, André. The future of mathematics. Amer. Math. Monthly **57**, 295–306 (1950).

Translated from a chapter in Les grands courants de la pensée mathématique, Cahiers du Sud, 1948; these Rev. **10**, 230.

*Sergescu, P. **Les recherches sur l'infini mathématique jusqu'à l'établissement de l'analyse infinitésimale.** Actualités Sci. Ind., no. 1083. Hermann et Cie., Paris, 1949. 32 pp.

This essay is an abstract of two lectures at the Institut Poincaré in Paris in May 1947. It deals with the history of the concepts of the calculus from the time of the Greeks to that of Newton and Leibniz. Particular attention is paid to the contribution of Archimedes, the School of Paris of the fourteenth century, Fermat, Cavalieri, Roberval, Wallis, Pascal, Newton, Leibniz and the Bernoulli brothers.

D. J. Struik (Cambridge, Mass.).

Taton, René. La préhistoire de l'analyse géométrique. Arch. Internat. Hist. Sci. (N.S.) **3**, 89–102 (1950).

Eilander, M. Historical remarks on the foundations of celestial mechanics. Simon Stevin **27**, 16–51 (1949). (Dutch)

Gloden, A. Aperçu historique des multigrades. Soc. Nat. Luxembourgeois. Bull. Mensuels. N.S. **43**, 18–23 (1949).

Gloden, A. Aperçu historique de la trigonométrie rectiligne et sphérique. Soc. Nat. Luxembourgeois. Bull. Mensuels. N.S. **43**, 1–17 (1949).

Hjelmslev, Johannes. Über Archimedes' Grössenlehre. Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 15, 13 pp. (1950).

This article deals with the exact meaning of the 5th Assumption in the Archimedean treatise "On the sphere and the cylinder," which is usually considered equivalent to the axiom of Eudoxos, i.e., Euclid V, Def. 4. The author stresses the necessity of distinguishing carefully between both. As the theory of Eudoxos does not apply to the domain of magnitudes considered by Archimedes, it needs an extension, which is carried into effect by the Assumption. This view is confirmed in the course of a discussion of the propositions "On the sphere and the cylinder" I, 2 and "On Spirals" 4, 12. *E. J. Dijksterhuis* (Oisterwijk).

*Morduhai-Boltovskoĭ, D. D. **Porisms and data.** Akademiya Nauk SSSR. Institut Istorii Estestvoznaniya. Trudy Soveščaniya po Istorii Estestvoznaniya 24–26 Dekabrya 1946 g., pp. 161–172. Izdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad, 1948. (Russian)

*Yuškevič, A. P. **On the ancient mathematicians' method of exhaustion.** Akademiya Nauk SSSR. Institut Istorii Estestvoznaniya. Trudy Soveščaniya po Istorii Estestvoznaniya 24–26 Dekabrya 1946 g., pp. 173–182. Izdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad, 1948. (Russian)

Rajagopal, C. T., and Venkataraman, A. The sine and cosine power-series in Hindu mathematics. J. Roy. Asiatic Soc. Bengal. Sci. **15**, no. 1, 13 pp. (1949).

The authors present the power-series for the sine and cosine as given in two Hindu mathematical works, the Tantrasaṅgraha and the Yukti-Bhāṣā. Of these two works, the Tantrasaṅgraha does not seem to have been published and the Yukti-Bhāṣā has recently been published in Malayāḷam. The question as to whether these series antedated their discovery in the West has been discussed by the authors, in an addendum by K. M. George, and in a special note by P. C. Sengupta. Sengupta properly observes that a thorough inquiry must be made into the dates of the Hindu works and into the possibility of foreign influence before any claims for the discovery of these series can be made. *E. B. Allen* (Troy, N. Y.).

Venkatraman, A. Some interesting proofs from "Yukti-Bhasha." Math. Student **16** (1948), 1–7 (1949).

Several proofs are given, in English and in modern notation with the necessary modifications, from the Malayāḷam edition of the Yukti-Bhāṣā. References to the Malayāḷam text are stated. The proofs are of the theorem of Pythagoras, the area of a triangle, three lemmas on cyclic quadrilaterals, and the area of a cyclic quadrilateral. These serve to give some idea of the contents of the Yukti-Bhāṣā to those who cannot read the Malayāḷam text. *E. B. Allen.*

Rajagopal, C. T. A neglected chapter of Hindu mathematics. Scripta Math. **15**, 201–209 (1949).

The method for establishing the ordinary power series for $\tan^{-1} t$, as contained in the Yukti-Bhāṣā, is first discussed. Following this, two rational approximations for $\pi/4$, taken from the same text, are given. The author formulates a theorem which leads to these approximations and notes that the Yukti-Bhāṣā uses the essential content of this theorem. *E. B. Allen* (Troy, N. Y.).

*Giacomelli, Raffaele. **Galileo Galilei giovane e il suo "De Motu."** Quaderni di Storia e Critica della Scienza, no. 1. Domus Galilaeana, Pisa, 1949. v+106 pp. 400 lire.

The well-known legendary stories about Galileo's youth and his lectureship at Pisa (the alleged discovery of the isochronism of the pendulum, the experiments on freely falling bodies from the Leaning Tower, his outspoken anti-Aristotelean attitude and the difficulties caused by it which forced him to leave Pisa) have been criticised several times by various authors outside Italy. Italian scholars, however, as a rule showed some irritability at these criticisms; in particular the late historian of science A. Milli would denounce as "detrattore" anyone not willing to swallow the traditional tales. The author of the present brochure takes a quite different attitude: not only does he share the critical mood as to the Galileo-legend which is current abroad, but he even corroborates it with new and convincing arguments. After having done so in the first chapter, in the second he makes a careful study of young Galileo's work De Motu [Opere, v. 1, pp. 243–419], which contains his ideas on weight, force and motion in the years at Pisa; these show a marked deviation from the theories developed in his maturer works, which makes them the more valuable for the investigation of the evolution of his scientific thinking. From among the various more or less erroneous statements on mechanical phenomena scattered throughout the treatise, the writer extracts some particular results in which germs of the future development of the science of motion are to be recognised. *E. J. Dijksterhuis* (Oisterwijk).

Agostini, Amedeo. L'uso delle lettere nel "Liber abaci" di Leonardo Fibonacci. Boll. Un. Mat. Ital. (3) **4**, 282–287 (1949).

Itard, Jean. Les méthodes utilisées par Fermat en théorie des nombres. Rev. Hist. Sci. Appl. **3**, 21–26 (1950).

Kuznecov, B. G. Absolute space in Euler's mechanics. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **1**, 347–371 (1947). (Russian)

Yuškevič, A. P. Euler and Russian mathematics in the XVIIIth century. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **3**, 45–116 (1949). (Russian)

Yuškevič, A. P. On the origin of Cauchy's definition of the integral. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **1**, 373–411 (1947). (Russian)

Yuškevič, A. P. Omar Khayyam and his "Algebra." Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **2**, 499–534 (1948). (Russian)

Costabel, P. Deux inédits de la correspondance indirecte Leibniz-Reyneau. Rev. Hist. Sci. Appl. **2**, 311–332 (1949).

Boyer, C. B. James Hume and exponents. Amer. Math. Monthly **57**, 7–8 (1950).

Grimm, G. Zur Kreisberechnung von Huygens. Elemente der Math. **4**, 78–85 (1949).

Valiron, Georges. Des théorèmes de Bloch aux théories d'Ahlfors. Bull. Sci. Math. (2) **73**, 152–162 (1949).

An account is given of the life and work of André Bloch. Particular reference is made to the relation between Bloch's work and subsequent function-theoretic research, especially the work of Ahlfors [Acta Soc. Sci. Fennicae. Nova Ser. A. **2**, no. 2 (1933); Acta Math. **65**, 157–194 (1935)]. *M. H. Heins* (Providence, R. I.).

Bržečka, V. F. On Bolzano's function. Uspehi Matem. Nauk (N.S.) **4**, no. 2(30), 15–21 (1 plate) (1949). (Russian)

A brief account of Bolzano's life and work with special emphasis on his well-known example of a continuous non-differentiable function. *F. A. Behrend* (Melbourne).

*Čebotarev, N. G. **Sobranie sočinenil. [Collected Works].** Izdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad, 1949. Vol. 1, ~~616 pp.~~ 543 pp. (1 plate); vol. 2, 588 pp.

Prudnikov, V. E. Academician P. L. Čebyšev and the Russian school. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **3**, 117–135 (1949). (Russian)

Prudnikov, V. E. New materials for the biography of P. L. Čebyšev. Uspehi Matem. Nauk (N.S.) **4**, no. 2(30), 173–175 (1949). (Russian)

Obituary: Émile Cotton. Ann. Inst. Fourier Grenoble **1** (1949), 1–4 (1 plate) (1950).

Mostowski, A. La vie et l'oeuvre de Samuel Dickstein. Prace Mat.-Fiz. **47**, VII–XII (1 plate) (1949).

Kubota, Tadahiko. Obituary note: Matsusaburô Fujiwara (1881–1946). Tôhoku Math. J. (2) **1**, 1–2 (1949).

Perron, Oskar. Obituary: Godfrey Harold Hardy. Jber. Bayer. Akad. Wiss. München **1944/48**, 282–285 (1948).

Littlewood, J. E., Pólya, G., Mordell, L. J., Titchmarsh, E. C., Davenport, H., and Wiener, Norbert. Two statements concerning the article on G. H. Hardy. Bull. Amer. Math. Soc. **55**, 1082 (1949).
Cf. Wiener, same vol., 72–77 (1949); these Rev. **10**, 420.

Petersson, Hans. Das wissenschaftliche Werk von E. Hecke. Abh. Math. Sem. Univ. Hamburg **16**, 7–31 (1949).

Perron, Oskar. Obituary: Erich Hecke. Jber. Bayer. Akad. Wiss. München **1944/48**, 274–276 (1948).

Perron, Oskar. Obituary: Kurt Hensel. Jber. Bayer. Akad. Wiss. München **1944/48**, 234–236 (1948).

Hasse, Helmut. Kurt Hensel zum Gedächtnis. J. Reine Angew. Math. **187**, 1–13 (1 plate) (1949).

Kneser, Hellmuth. Felix Klein. Zu seinem hundertsten Geburtstage am 25. April 1949. Arch. Math. **1**, 413–417 (1949).

***Kočin, N. E. Sobranie sočinenii. [Collected Works].** Izdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad, 1949. Vol. 1, 344 pp.; vol. 2, 420 pp.
The first volume contains papers on dynamical meteorology; the second, papers on hydrodynamics and aerodynamics, applied and theoretical mechanics, and mathematics. One paper previously not available and one posthumous paper will be reviewed separately in these Rev.

The list of mathematical papers published by Prof. T. Kubota. Tôhoku Math. J. (2) **1**, 3–13 (1949).

Venkov, B. A., and Natanson, I. P. Obituary: Rodion Osievič Kuz'min (1891–1949). Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 148–155 (1 plate) (1949). (Russian)

Smirnov, V. I. Obituary: Rodion Osievič Kuz'min. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 385–388 (1949). (Russian)

***Lobačevskiĭ, N. I. Polnoe sobranie sočineniĭ. Tom četvertyĭ. Sočineniya po algebre. [Complete Collected Works. Volume Four. Works on Algebra].** Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad, 1948. 472 pp.

***Modzalevskiĭ, L. V. Materialy dlya biografii N. I. Lobačevskogo. [Materials for a Biography of N. I. Lobačevskiĭ].** Izdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad, 1948. 827 pp.

Depman, I. Ya. New facts about N. I. Lobačevskiĭ. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **2**, 561–563 (1948). (Russian)

Lunc, G. L. The analytic work of N. I. Lobačevskiĭ. Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 187–195 (1950). (Russian)

***Lyapunov, A. M. Izbrannye trudy. [Selected Works]. Edited by V. I. Smirnov, with notes by S. N. Bernšteĭn, L. N. Sretenskiĭ and N. G. Četaev.** Izdatel'stvo Akademii Nauk SSSR, 1948. 540 pp.

***Papaleksi, N. D. Sobranie trudov. [Collected Works]. Edited by S. M. Rytov.** Izdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad, 1948. 428 pp.

***Petrović, Mihailo. Članci. [Papers].** Naučna Knjiga, Belgrade, 1949. viii+112 pp.

Plá, Cortés. The transcendent work of Father Mersenne. Actas Acad. Ci. Lima **12**, 44–57 (1949). (Spanish)

Blaquier, Juan. Sir Isaac Newton. The man and the mathematician. Anales Acad. Nac. Ci. Ex. Fís. Nat. Buenos Aires **12**, 9–32 (1947). (Spanish)

Rossinskiĭ, S. D. Obituary: Karl Mihallovič Peterson (1828–1881). Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 3–13 (1 plate) (1949). (Russian)

Perron, Oskar. Obituary: Alfred Pringsheim. Jber. Bayer. Akad. Wiss. München **1944/48**, 187–193 (1948).

Archibald, R. C. Rheticus, with special reference to his Opus Palatinum. Math. Tables and Other Aids to Computation **3**, 552–561 (1949).

Gloden, A. A propos d'un quadricentenaire. Simon Stevin, mécanicien et mathématicien (1548–1620). Soc. Nat. Luxembourgeois. Bull. Mensuels. N.S. **42**, 70–73 (1948).

Gloden, A. La vie et l'oeuvre scientifique de neuf mathématiciens belges d'origine luxembourgeoise. Thémecht **2**, 12–36 (1949).
The mathematicians in question are A. Meyer, J.-B. Brasseur, M. Steichen, M. Schaar, J. Neuberg, J.-P. Schmit, P. Schorn, N. Breithof, N. Sibenaler.

Behnke, Heinrich. Otto Toeplitz zum Gedächtnis. Math.-Phys. Semesterber. **1**, 89–96 (1949).

Aleksandrov, P. S. Obituary: Pavel Samuilovič Uryson. Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 196–202 (1 plate) (1950). (Russian)

***VI Polski Zjazd Matematyczny. Jubileusz 40-Lecia Działalności na Katedrze Uniwersyteckiej Profesora Wacława Sierpińskiego. Warszawa, 23.9.1948.** Staraniem Komitetu Jubileuszowego, Warszawa, 1949. 95 pp.
This volume was issued as a supplement to Ann. Soc. Polonaise Math. **21**. It contains among other things a complete list of Sierpiński's publications.

Andronov, A. A. I. A. Vyšnegradskiĭ and his role in the creation of the theory of automatic regulation. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 805–818 (1949). (Russian)

Crespo Pereira, Ramón. Obituary: Alfred North Whitehead. Revista Mat. Hisp.-Amer. (4) **9**, 49–52 (1949). (Spanish)

Carathéodory, C. Obituary: Wilhelm Wirtinger. Jber. Bayer. Akad. Wiss. München **1944/48**, 256–258 (1948).

***Žukovskiĭ, N. E. Izbrannye sočineniya. [Selected Papers].** OGIZ, Moscow-Leningrad, 1948. Vol. 1, 391 pp.; vol. 2, 422 pp.

ALGEBRA

Levy, Paul. Sur quelques classes de permutations. Compositio Math. **8**, 1–48 (1950).

Proofs of results previously reported [C. R. Acad. Sci. Paris **227**, 422–423, 578–579 (1948); these Rev. **10**, 177, 130], and some additional material, notably two tables giving the cycle decomposition of the two kinds of permutation for all $n<46$, and scattered results for interesting n above this. *J. Riordan* (New York, N. Y.).

Jacobsthal, Ernst. Über die Anzahl von Permutationen von n Elementen, die r Inversionen besitzen. I. Norske Vid. Selsk. Forh., Trondheim **22**, no. 10, 31–36 (1950).

Known recurrence formulas [E. Netto, Lehrbuch der Kombinatorik, Teubner, Leipzig, 1901, pp. 92–97] for the number of permutations of n elements with r inversions are derived. *J. Riordan* (New York, N. Y.).

Jacobsthal, Ernst. Über die Anzahl von Permutationen von n Elementen, die r Inversionen besitzen. II. Norske Vid. Selsk. Forh., Trondheim **22**, no. 11, 37–41 (1950).

The generating function for permutations of n elements with r inversions, due to O. Rodrigues [J. Math. Pures Appl. (1) **4**, 236–240 (1839)], is redetermined and used to give a determinantal expression having terms involving $\sigma(m, n)$, $m=1$ to r, the sum of divisors of m not greater than n, for the numbers of such permutations, $\psi(n, r)$, and also the formula $\psi(n, r)=\sum_k \psi(r, k)\binom{n-1-k}{r-k}$, $n \geqq r$. *J. Riordan* (New York, N. Y.).

Banerjee, K. S. On the variance factors of weighing designs in between two Hadamard matrices. Calcutta Statist. Assoc. Bull. **2**, 38–42 (1949).

This computes the inverse of GG', where G is a Hadamard matrix H with two or three different rows of H appended to H. *A. M. Mood* (Santa Monica, Calif.).

Mattioli, Ennio. Segnatura e divisori elementari di una matrice. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 328–336 (1949).

The author establishes a connection between Cherubino's canonical form of a matrix [see Boll. Un. Mat. Ital. (2) **4**, 38–48 (1942); these Rev. **7**, 358] and the theory of elementary divisors. *O. Todd-Taussky* (Washington, D. C.).

Bell, James H. Families of solutions of the unilateral matrix equation. Proc. Amer. Math. Soc. **1**, 151–159 (1950).

The unilateral matrix equation $R(X)=\sum_{m=0}^{r} R_m X^m=0$ is studied over a field of characteristic zero, where the R_m and X are square of order n, with particular reference to the case in which there exist an infinite number of solutions. The method is based on a detailed study of the factorization $R[\lambda]=P[\lambda]A[\lambda]$, where $R[\lambda]=\sum_{m=0}^{r} R_m\lambda^m$ and R and A are put in a canonical triangular form of Hermite type. If a set of right factors of R have the same main diagonal and if these right factors are left associates of matrices of the form $\lambda I-X$, the matrices X so obtained are called a family of solutions of $R(X)=0$. It is shown that an infinite family of solutions exists if and only if there exist two distinct similar solutions. All matrices similar to a given solution are solutions if and only if their minimal polynomial is a common divisor of the elements of $R[\lambda]$. If $R(X)=0$ has a solution and $\det(R[\lambda])\equiv 0$, there exist an infinite number of solutions, which need not form a family in the above sense. *W. Givens* (Knoxville, Tenn.).

Aranovič, B. I. The use of matrix methods in problems of the structural analysis of relay-contact schemes. Avtomatika i Telemehanika **10**, 437–451 (1949). (Russian)

Let the nodes of a switching circuit be enumerated $1, 2, \cdots, n$. Associated with the circuit is a matrix $C=(c_{ij})$ where $c_{ii}=1$ and c_{ij} $(i \neq j)$ is the structural formula representing the switch or group of switches in parallel joining nodes i and j. The formula for determining whether or not there is a closed path between nodes 1 and 2 is obtained by deleting row 1 and column 2 from C and formally writing out the determinant of the remaining matrix (ignoring minus signs).

If the circuit element joining nodes i and j is a rectifier which allows current to flow only from i to j, set $c_{ij}=1$ and $c_{ji}=0$. By evaluating determinants one can find the switch settings which permit current flow from 1 to 2 and those which permit flow from 2 to 1. *E. N. Gilbert.*

Lunc, A. G. Application of matrix Boolean algebra to the analysis and synthesis of relay-contact schemes. Doklady Akad. Nauk SSSR (N.S.) **70**, 421–423 (1950). (Russian)

Let the circuit element joining nodes i and j in a switching network be governed by a Boolean function a_{ij}. Let the Boolean function which determines when there is a closed path through the network from i to j be $\chi_{ij}(A)$. These functions are made elements of two matrices A and $\chi(A)$. Two methods of network analysis (determination of $\chi(A)$ from A) are given. One is the method of evaluating determinants which is also described in the paper reviewed above. Another method consists in computing powers of A. One finds $A^r=A^{r+1}=\cdots$ for some r less than the number of nodes; then $\chi(A)=A^r$. Two matrices A and A' represent equivalent circuits if $\chi(A)=\chi(A')$. It is shown that, if $\chi(A)$, $\chi(X)$, and $\chi(Y)$ satisfy certain order requirements, A and $A\bar{X}+Y$ are equivalent. Here $\bar{X}$ is the matrix whose off-diagonal terms are the inverses of the elements of X. *E. N. Gilbert* (Murray Hill, N. J.).

Gavrilov, M. A. Concerning the analysis of relay-contact schemes. Doklady Akad. Nauk SSSR (N.S.) **69**, 181–184 (1949). (Russian)

The paper contains examples of the use of expansion theorems on Boolean functions in the analysis of switching networks. *E. N. Gilbert* (Murray Hill, N. J.).

Platone, Giulio. Complementi al metodo delle parabole osculanti per la separazione delle radici di una equazione di quarto grado. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 43–51 (1949).

Let $f(x)$ be a polynomial and $g(x, x_0)$ its Taylor expansion about x_0 as far as the quadratic term. The curve Γ: $y=g(x, x_0)$ is the osculating parabola of the curve C: $y=f(x)$ at the point x_0. If $f(x)$ is the quartic $x^4+a_1x^3+\cdots$, then Γ intersects C in the points x_0 and $-a_1-3x_0$. These points coincide if and only if x_0 is the root ξ of f''' and the corresponding parabola $y=g(x, \xi)$ has been employed by Picone to separate the roots of a quartic whose graph is concave upward. If $x_0 \neq \xi$, curves Γ and C intersect in distinct points whose projections on the x-axis are separated by ξ and define the segment on which the parabola is above the quartic. The present paper bases on this property a method for separating the real roots applicable to all cases of the quartic. Incidentally, it is remarked that the bitangent to the quartic, the join of its points of inflexion and the tangent at the root of f''' all have the same direction.

J. M. Thomas (Durham, N. C.).

Checcucci, Vittorio. Sulle omografie che trasformano in sè una quadrica o una antiquadrica dell' S_{n-1}. Ann. Scuola Norm. Super. Pisa (3) **2** (1948), 131–149 (1950).

Die Arbeit knüpft an Untersuchungen von G. Frobenius [J. Reine Angew. Math. **84**, 1–63 (1877)], C. Segre [Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (2) **37**, 395–425 (1886)], G. Dantoni [Ann. Scuola Norm. Super. Pisa (2) **6**, 11–28 (1937)] und S. Cherubino [Rend. Sem. Mat. Univ. Roma (4) **1**, 96–109 (1936); 139–174 (1937)] an. Seien A, X usw. nicht singuläre Matrizen mit reellen oder komplexen Elementen. Verfasser bestimmt zunächst alle Matrizen X, die zu gegebenem A der Gleichung genügen $A_{-1}XA=X$, wo A_{-1} die transponierte Matrix zu A ist; insbesondere interessieren hier die symmetrischen bzw. schiefsymmetrischen Lösungen X. Analog wird die Gleichung $A_{-1}X\bar{A}=X$ untersucht, wo $\bar{A}$ die konjugiert-komplexe Matrix zu A ist. Als geometrische Anwendungen dieser Untersuchungen gewinnt Verfasser einige der von den obengenannten italienischen Autoren abgeleiteten Sätze wieder: sie betreffen z.B. das Verhalten der linearen Mannigfaltigkeiten grösster Dimensionszahl auf einer quadratischen Mannigfaltigkeit Q_{n-1} ungerader Dimensionszahl bei einer linearen Transformation von Q_{n-1} in sich; ferner eine nicht singuläre Korrelation B des S_{n-1} in sich, die mit einer Kollineation A durch die Bedingung $B=\alpha A_{-1}BA$ (α ein Proportionalitätsfaktor) verknüpft ist; endlich Sätze über das Verhalten gewisser ausgezeichneter linearer Mannigfaltigkeiten, die der Matrix A vermöge $(A-\rho E)\cdot x_{-1}=0$ (x_{-1} Zeilenvektor, ρ charakteristische Wurzel von A) zugeordnet sind und der dazu dualen Gebilde bei Korrelationen, die mit einer Kollineation vertauschbar sind.

R. Moufang (Frankfurt am Main).

Abstract Algebra

Büchi, J. Richard. Die Boole'sche Partialordnung und die Paarung von Gefuegen. Portugaliae Math. **7**, 119–180, 207 (1948).

The author seeks an axiomatic which will bear the same relation to Boolean algebra as partial ordering does to lattice theory. Then mappings of non-atomic orderings are studied, leading to "pairing," a generalization of join-homomorphism. A relation "$\subset$" on B_0 is a Boolean partial order if, under $\subset$, B_0 is partially ordered with smallest element 0 and $x \not\subset y$ implies that there exists $u \neq 0$ with $u \subset x$ and uoy, where o is defined by wox if and only if $u \subset w$ and $u \subset z$ imply $u \subset 0$. Equivalently, B (B_0 with 0 deleted) satisfies the postulates: xoy and $y \subset x$ are never both true; xoy and $z \subset y$ imply xoz; if uoy implies uox for all u then $x \subset y$; xoy if for all u, $u \subset x$ or $u \not\subset y$; then B is called a "Gefüge." The order in a generalized Boolean algebra is a Boolean partial order. A Boolean partially ordered set B_0 is a complete Boolean algebra if and only if distributive unions always exist. Any Boolean partially ordered set can be imbedded in a minimal complete Boolean algebra preserving order and distributive joins. Any atomic Boolean partially ordered set is isomorphic to the class of all subsets of its atoms. Bases of topological sets are considered in the light of these ideas.

Given Boolean partially ordered sets B_0, B_0^*, and single-valued mappings f, f^*, of B_0 into B_0^* and vice versa, such that $vof(u)$ if and only if $uof^*(v)$ for all $u \in B_0$, $v \in B_0^*$; then (ff^*) is a pairing of B_0, B_0^*. If B_0, B_0^* are complete Boolean algebras and f a map of B_0 into B_0^*, then f is a homomorphism with respect to unrestricted joins if and only if it is a member of a pairing.

P. M. Whitman.

Lombardo Radice, Lucio. Su alcune catene massime della struttura distributiva libera con n generatori. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 126–133 (1949).

It is known that the free distributive lattice with n generators has dimension 2^n. The author obtains explicitly a maximal chain. [For a discussion of this lattice, see T. Skolem, Tredje Skandinaviske Matematikerkongres i Kristiania 1913, pp. 149–163 (1915); G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, revised ed., New York, 1948, pp. 145–146; these Rev. **10**, 673.]

P. M. Whitman (Silver Spring, Md.).

Krishnan, Viakalathur S. L'extension d'une $(<, \cdot)$ algèbre à une $(\sum^*, \cdot)$ algèbre. C. R. Acad. Sci. Paris **230**, 1447–1448 (1950).

Let P be a partially ordered set closed under an order-preserving binary operation $(\cdot)$. Then P can be extended to an algebra closed under distributive sums, and in a sense [details and proofs to be published later] there is a unique minimal such extension.

P. M. Whitman.

Lyapin, E. S. Normal complexes of associative systems. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 179–192 (1950). (Russian)

The author continues his study of associative systems [Rec. Math. [Mat. Sbornik] N.S. **20**(62), 497–515 (1947); these Rev. **9**, 134], no longer assuming a unit element. Normal subsystems are defined as by Siverceva [Mat. Sbornik N.S. **24**(66), 101–106 (1949); these Rev. **10**, 508]. A normal complex K in a system A is a subset such that for $x, y \in A$ and $k, k' \in K$, $xky \in K$ implies $xk'y \in K$, $xk \in K$ implies $xk' \in K$, $ky \in K$ implies $k'y \in K$. In a homomorphism, the inverse images of elements are normal complexes, and conversely for any normal complex there exists a corresponding homomorphism. If the homomorphism is onto a system with unit or zero, respectively, the kernel is a normal subsystem or ideal, respectively. If $\{A_i\}$ are systems with 0, their "mutually annihilating" sum is defined by taking the set-theoretic union, preserving the product in each A_i, and

making the product 0 between different A's. Every system with 0 has a unique decomposition of this kind into indecomposable parts. The normal complexes, subsystems and ideals of a mutually annihilating sum are studied.

I. Kaplansky (Chicago, Ill.).

Abellanas, Pedro. Clarification of the article "Orderable fields with a single automorphism." Revista Mat. Hisp.-Amer. (4) **9**, 80 (1949). (Spanish)

[See the same vol., 3–9 (1949); these Rev. **11**, 5.] The author's theorem now reads that conditions (a) and (b) (of the cited review) are sufficient for a field to allow only the identical automorphism, and that if either condition fails K may have a nontrivial automorphism.

D. B. Scott (London).

Ratnam, Perala. Algebraic functions. Kōdai Math. Sem. Rep., no. 5–6, 23–38 (1949).

This is a thesis in which the local behavior of the defining equation of an algebraic function field of one variable over an algebraically closed field of characteristic zero is worked over in detail. The note terminates with a proof of the Riemann-Roch theorem.

O. F. G. Schilling.

Weissinger, Johannes. Zur arithmetischen Theorie separierbarer Funktionenkörper. Abh. Math. Sem. Univ. Hamburg **16**, 155–163 (1949).

Let K be a field of algebraic functions of one variable separable over its field of constants k, and set $K=k(x, y)$, with y separable over $k(x)$. The author first gives a new proof of the fact that $(dx/dy)=\mathfrak{B}_x\mathfrak{N}_x^{-2}\mathfrak{B}_y^{-1}\mathfrak{N}_y^2$, where $\mathfrak{B}_x$ and $\mathfrak{B}_y$ are the ramification divisors of x and y, and $\mathfrak{N}_x$ and $\mathfrak{N}_y$ their denominators. The proof is made by going over to the $\mathfrak{P}$-adic closure $K_\mathfrak{P}$ of K, where $\mathfrak{P}$ is any place of K. Let $\chi(y)$ be the irreducible polynomial of y over the $\mathfrak{P}$-adic closure $K(x)_\mathfrak{P}$ of $K(x)$ and $\mathfrak{d}_y$ the different of $K_\mathfrak{P}$ over $K(y)_\mathfrak{P}$; let also $\mathfrak{f}$ be the conductor of the ring of integers of $K_\mathfrak{P}$ over the $\mathfrak{P}$-adic closure of $k[x, y]$; then it is proved by rather straightforward computation that $(\partial\chi/\partial x)$ is a multiple of $\mathfrak{f}\mathfrak{d}_y$ (in all this, x and y are assumed to be integral at $\mathfrak{P}$); the result then follows easily. The second question considered in the paper is the effect of an extension k'/k of the basic field. It is shown that the degrees of the divisors of K are invariant, and that two divisors of K which are equivalent in $K'=k'K$ are already equivalent in K. An example shows that the genus may be diminished by the extension. Let $\mathfrak{O}$ and $\mathfrak{O}'$ be the rings of integers of K and K' with respect to $k[x]$ and $k'[x]$ respectively; then it is shown that the conductor $\mathfrak{F}$ of $\mathfrak{O}'$ over $\mathfrak{O}[k']$ does not depend on x. The ramification divisors $\mathfrak{B}_x$ and $\mathfrak{B}_x'$ of x in K and K' are related to each other by the formula $\mathfrak{B}_x=\mathfrak{F}\mathfrak{B}_x'$.

C. Chevalley (New York, N. Y.).

Hasse, Helmut. Invariante Kennzeichnung galoisscher Körper mit vorgegebener Galoisgruppe. J. Reine Angew. Math. **187**, 14–43 (1949).

This paper contains a generalization of the work of the author on the characterization of Abelian normal rings over a field Ω [see, for example, Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1947**, no. 8 (1949); these Rev. **11**, 155]. If K/Ω is an Abelian extension of degree n with the Galois group $G=\{S, \cdots\}$ and character group $X=\{\chi, \psi, \cdots\}$, then K/Ω has a "factor basis" $\{\omega_\chi\}$ of radicals with $\omega_\chi^S=\chi(S)\omega_\chi$. Such a factor basis is related to a normal basis $\{\theta^S, S\in G\}$ of K/Ω by the equations $\omega_\chi=(1/n)\sum_S\chi(S^{-1})\theta^S$. Noting that an existence proof for normal bases may be given by means of E. Noether's and M. Deuring's theory of Galois modules [see Noether, same J. **167**, 147–152 (1932); Deuring, Math. Ann. **113**, 40–47 (1936)], the author discusses the following generalization of the Abelian theory.

(I) If Ω contains the nth roots of unity then the group ring $G\times\Omega$ is the direct sum of matrix algebras $(G\times\Omega)e_\chi$, with idempotents e_χ, of respective degrees $f_\chi=\chi(1)$, each corresponding to a simple character χ of G, provided n is relatively prime to the characteristic of Ω [see R. Brauer, Amer. J. Math. **69**, 709–716 (1947); these Rev. **9**, 268]. Each simple right ideal G_χ^k of $(G\times\Omega)e_\chi$ gives rise to an absolutely irreducible representation Γ_χ of G belonging to the simple character χ. If $Se_\chi=\sum_{i,k}a_\chi^{ik}(S)e_\chi^{ik}$ is the system of equations expressing the effect of S upon the idempotent e_χ in terms of a system of matrix units $(e_\chi^{ik})=E_\chi$ of $(G\times\Omega)e_\chi$, then $E_\chi S=A_\chi(S)E_\chi$ where the representing matrix $A_\chi(S)$ lies in the system $\{\Gamma_\chi\}$ of equivalent representations belonging to χ. The author collects in this connection these and other orthogonality and reciprocity relations of Frobenius' theory of characters. Next the representation ring of G, where direct sum and Kronecker product are the basic operations, is described in detail. Let thus $\Gamma_\chi\times\Gamma_\psi=P_{\chi,\psi}^{-1}\Gamma_{\chi\psi}P_{\chi,\psi}$ with $\Gamma_{\chi\psi}=\operatorname{diag}_t(I^t{}_{\chi\psi}\times\Gamma_t)$ where $P_{\chi,\psi}$ is a suitable regular matrix of degree $f_\chi f_\psi$ in Ω and where Γ_t occurs $k^t{}_{\chi\psi}$ times. It is to be noted that the algebra of the representation ring implies that $P_{\chi,\psi}$ may be replaced by $\operatorname{diag}_t(A^t{}_{\chi\psi}\times I_t)P_{\chi,\psi}$ where the matrices $A^t{}_{\chi\psi}$ are arbitrary of degrees $k^t{}_{\chi\psi}$ in Ω. Furthermore there are the relations expressing that the representation ring is commutative and associative, i.e., $\Gamma_\chi\times\Gamma_\psi$, $\Gamma_\psi\times\Gamma_\chi$, and $\Gamma_\chi\times(\Gamma_\psi\times\Gamma_\varphi)$, $(\Gamma_\chi\times\Gamma_\psi)\times\Gamma_\varphi$ belong to the same classes of representations of G, respectively.

(II) The field K/Ω may be considered as a "Galois module," i.e., an Ω-module with the elements $S\in G$ for right operators. Then K/Ω is isomorphic to the group ring $G\times\Omega$ with respect to $G\times\Omega$ for a common ring of right operators. Thus, each $(G\times\Omega)$-isomorphism ι between $G\times\Omega$ and K determines, according to E. Noether and M. Deuring [loc. cit.], in K a set of elements $\theta^S=\iota(S)$ which form a normal basis of K/Ω. Now it is most important that the representation $S\to R(S)$ where $(\theta^T)^S=R(S)(\theta^T)$ is equivalent to the regular representation of G, i.e., it contains for its constituents all absolutely irreducible representations Γ_χ of G.

(III) After this remark the isomorphism between $G\times\Omega$ and K/Ω is applied to the simple two-sided ideals $(G\times\Omega)e_\chi$ and the simple right ideals G_χ^k according to (I). Therefore there corresponds to E_χ a matrix W_χ of elements of K with $W_\chi{}^S=A_\chi(S)W_\chi$ where $W_\chi=(1/n)\sum_S f_\chi A_\chi(S^{-1})\theta^S$ with the particular representation A_χ in the class $\{\Gamma_\chi\}$ of the simple character χ. Once this observation is made the formal analogue to the factor basis (Lagrange resolvent) of the Abelian theory is established. Thereupon the author proceeds in the obvious fashion: The effect of a change of the isomorphism ι, the changes $W_\chi\to W_\chi A_\chi$ with regular A_χ of Ω, the commutativity and associativity relations between the W_χ, which correspond to those mentioned under (I), are derived. Thus, for example, $W_\chi\times W_\psi=P_{\chi,\psi}^{-1}W_{\chi\psi}P_{\chi,\psi}\cdot C_{\chi,\psi}$ with a factor set of $f_\chi f_\psi$-dimensional matrices in Ω, with the equivalence relation $C_{\chi,\psi}\to(P_{\chi,\psi}^{-1}\operatorname{diag}_t(I^t_{\chi\psi}\times A_t)P_{\chi,\psi})C_{\chi,\psi}(A_\chi\times A_\psi)$ if $W_\chi\to W_\chi A_\chi$, are found. Keeping in mind the effect of the changes $\Gamma_\chi\to B_\chi{}^{-1}\Gamma_\chi B_\chi$ in the representation classes $\{\Gamma_\chi\}$, it follows that K/Ω determines uniquely a class C of factor sets of matrices $(C_{\chi,\psi})$ subject to the further equivalence relation $C_{\chi,\psi}\to(B_\chi\times B_\psi)^{-1}C_{\chi,\psi}(B_\chi\times B_\psi)$. It is to be noted that all these results do not require that K/Ω is a field with the group G. It suffices that K/Ω is a normal ring with G

for a group of automorphisms, i.e., (i) K is a commutative and associative algebra over Ω with the elements of G for operators, and (ii) K/Ω is $(G\times\Omega)$-right-isomorphic to $G\times\Omega$. Finally, the author shows that, conversely, each class of factor sets C of G in Ω gives rise to a normal ring K/Ω if matrices W_x composed of quantities over Ω are introduced between which the rules of (I), (II) and (III) are postulated for the definition of the algebraic operations.

O. F. G. Schilling (Chicago, Ill.).

Barsotti, Iacopo. Valutazioni nelle algebre divisorie senza base finita. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 168–185 (1949).

Let g be a Dedekind ring (integral domain in which every ideal is a unique product of prime ideals). Let A be a division algebra (no assumption of finite order) over the quotient field F of g. With the usual definitions, it is shown that any g-array is contained in a maximal g-array. Let H be the multiplicative commutator subgroup of A. Then any valuation of A which is integral on g leads to a maximal g-array containing H; and conversely any maximal g-array containing H is attached to an essentially unique valuation of A. Conditions for the existence of such a valuation are given. In the second part of the paper, F is specialized to be a local field (completion of an algebraic number field), and it is assumed that every two elements of A generate a finite-dimensional subalgebra. Then the valuation of F extends uniquely to A. Residue class degree and ramification order can be defined as Steinitz G-numbers, and many of the known facts of the finite-dimensional case have satisfactory generalizations. [There is some slight overlapping with a paper of Schilling: Bull. Amer. Math. Soc. **51**, 297–304 (1945); these Rev. **6**, 201.]

I. Kaplansky (Chicago, Ill.).

Chevalley, Claude, and Schafer, R. D. The exceptional simple Lie algebras F_4 and E_6. Proc. Nat. Acad. Sci. U. S. A. **36**, 137–141 (1950).

Soit $\mathfrak{C}$ l'algèbre (non associative) des octaves de Cayley sur un corps K algébriquement clos et de caractéristique 0; on sait depuis la thèse de É. Cartan que les dérivations de $\mathfrak{C}$ forment une algèbre de Lie qui est isomorphe à l'algèbre simple exceptionnelle G_2 de dimension 14. Dans cette note, les auteurs indiquent des relations entre $\mathfrak{C}$ et les algèbres simples exceptionnelles F_4 (dimension 52) et E_6 (dimension 78); les résultats sont les suivants. Soit $\mathfrak{J}$ l'espace des matrices hermitiennes d'ordre 3 sur $\mathfrak{C}$, espace muni de la loi de composition $X\circ Y=\frac{1}{2}(XY+YX)$: on obtient ainsi l'algèbre de Jordan simple exceptionnelle sur K; ceci étant, F_4 est isomorphe à l'algèbre des dérivations de $\mathfrak{J}$; par ailleurs, pour $X\in\mathfrak{J}$, soit R_X l'endomorphisme $Y\to Y\circ X$ de $\mathfrak{J}$; alors l'algèbre de Lie engendrée par les dérivations de $\mathfrak{J}$ et les opérateurs R_X associés aux $X\in\mathfrak{J}$ dont la trace (dans $\mathfrak{J}$) est 0 est isomorphe à E_6. La démonstration contient évidemment d'assez nombreux calculs (y compris des additions, portant sur des nombres de paramètres: mais ce n'est pas le point le plus difficile), et aussi une formulation du principe de trialité, que voici. Pour $x\in\mathfrak{C}$, posons $\operatorname{Sp} x=x+\bar{x}$; considérons d'autre part l'algèbre de Lie $O(8,K)$ formé des opérateurs U définis dans $\mathfrak{C}$ tels que $(Ux)\bar{x}+x(\overline{Ux})=0$; alors pour tout $U\in O(8,K)$ il existe deux éléments U', U'' de $O(8,K)$, bien déterminés, tels que l'on ait $\operatorname{Sp}(Ux)yz+\operatorname{Sp}x(U'y)z+\operatorname{Sp}xy(U''z)=0$ quels que soient x, y, $z\in\mathfrak{C}$.

R. Godement (Nancy).

Wessel, Walter. Zur relativistischen Quantenmechanik. Z. Naturforschung **4a**, 645–653 (1949).

The author constructs and classifies representations of a certain Lie algebra from the representations of the Lorentz group given by Harish-Chandra [Proc. Roy. Soc. London. Ser. A. **189**, 372–401 (1947); these Rev. **9**, 132]. This Lie algebra is one that the author has discussed in a previous paper [Physical Rev. (2) **76**, 1512–1519 (1949); these Rev. **11**, 318] and it contains the generators of the Lorentz group as a subalgebra. The paper contains some remarks on the construction of a Hamiltonian in terms of elements of the algebra.

A. H. Taub (Urbana, Ill.).

Wang, Shianghaw. On the commutator group of a simple algebra. Amer. J. Math. **72**, 323–334 (1950).

Nakayama and Matsushima proved [Proc. Imp. Acad. Tokyo **19**, 622–628 (1943); these Rev. **7**, 238] that in every simple algebra over a p-adic number field, every element of reduced norm 1 is in the commutator group (i.e., the group of elements $\alpha\beta\alpha^{-1}\beta^{-1}$). The present paper shows that this is also true for simple algebras over an arbitrary field, if their index over the center is square-free, and for arbitrary simple algebras over an algebraic number field. The proof of the last fact makes extensive use of the author's theorem on cyclic fields with given local components [Ann. of Math. (2) **51**, 471–484 (1950); these Rev. **11**, 489].

The paper also contains a proof that if A and A' are the group of regular elements, and the commutator group, of a 2×2 matrix algebra over a division ring $\mathfrak{D}$, then all proper normal subgroups of A/A' are in the center whenever $\mathfrak{D}$ is not the Galois field of order 2 or 3. This is an addition to results of Dieudonné [Bull. Soc. Math. France **71**, 27–45 (1943); these Rev. **7**, 3].

G. Whaples.

Rees, D. The nuclei of non-associative division algebras. Proc. Cambridge Philos. Soc. **46**, 1–18 (1950).

The three nuclei $\mathfrak{N}_\lambda$, $\mathfrak{N}_\mu$, $\mathfrak{N}_\rho$ of a non-associative algebra are those sub-algebras which Bruck has defined as the left-, middle-, and right-associators, respectively [Trans. Amer. Math. Soc. **60**, 245–354 (1946); these Rev. **8**, 134]. Structure theorems are established for algebras satisfying certain conditions on the intersections of the nuclei. A principal result is the following. Let $\mathfrak{A}$ be a central division algebra of order n over $\mathfrak{F}$ such that $\mathfrak{N}_\lambda\cap\mathfrak{N}_\mu\cap\mathfrak{N}_\rho$ contains a subfield $\mathfrak{B}$ normal of order t over $\mathfrak{F}$. Then $n=kt^2$, k a positive integer, and we can find kt elements $u_{T,i}$ ($T\in G$, the automorphism group of $\mathfrak{B}$, $i=1,\cdots,k$) such that every element in $\mathfrak{A}$ is expressible uniquely in the form $\sum_{T,i}x_{T,i}u_{T,i}$ ($x_{T,i}\in\mathfrak{B}$), and such that multiplication in $\mathfrak{A}$ is determined by the distributive law and the equations

$$(xu_{T,i})(yu_{U,j})=\sum_l\gamma_{ij}^{T,U}x\cdot y^T\cdot u_{UT,l},$$

where the t^2k^3 quantities $\gamma_{ij}^{T,U}$ belong to $\mathfrak{B}$.

F. Kiokemeister (South Hadley, Mass.).

THEORY OF GROUPS

Todd, J. A. The invariants of a finite collineation group in five dimensions. Proc. Cambridge Philos. Soc. **46**, 73–90 (1950).

As a sequel to recent work by Baker [A Locus with 25920 Linear Self-Transformations, Cambridge University Press, 1946; these Rev. **8**, 400] and Todd [Proc. Roy. Soc. London. Ser. A. **189**, 326–358 (1947); these Rev. **9**, 6] on a collineation group G^* of order 25920 in S_4, the present paper is an investigation of the invariants of a related but more ample collineation group G of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$ in S_5. Both groups were noted by Mitchell [Amer. J. Math. **36**, 1–12 (1914)] as being among those which can be generated by homologies; and Mitchell showed that G contains 126 homologies, that the axial prime of any one of these contains the vertices of 45 others, and that these 45 generate a collineation group G^* in the prime in question. Recently C. M. Hamill [in a paper not yet published] has described in detail the character of the operations of G and their distribution into conjugate sets. These results are used here by Todd in investigating the invariants of G.

In § 1 the author compares the classical problem of finding the invariants of a finite group of linear homogeneous substitutions with that of finding the primals which are invariant under a finite group of collineations. To deal with the latter problem he gives an explicit construction for associating a finite linear substitution group Γ with any given collineation group G. To find the invariant primals of G it is then necessary to examine not only the invariants of Γ but also those of any invariant subgroups of Γ whose factor groups are cyclic. Any invariant of such a group Γ_1 will correspond to a primal which is invariant either under G or under an invariant subgroup of G which contains the collineation group G_1 corresponding to Γ_1; and the decision between these alternatives must be made by examining the effect of the collineations of G on the primal in question.

In § 2 he summarizes the configurational properties of the particular group G to be considered. In § 3 he constructs, by the method of § 1, an associated substitution group Γ of order $6N$, where N is the order of G. This contains only one invariant subgroup Γ_1, of index 2, whose factor group is cyclic, this corresponding to the invariant subgroup G_1 of G which consists of those operations of G which are products of an even number of homologies. In § 4 he considers each of the 31 types of operations in G (as enumerated by Hamill), each type consisting of one or two complete sets of conjugate operations; and he constructs for each type its total contribution to the generating function of Γ. In § 5, knowing the generating functions of Γ and Γ_1, he infers the probable existence of irreducible invariant primals of orders 6, 12, 18, 24, 30, and 126. Finally in § 6 he obtains four forms of the equation of the sextic invariant primal $I_6=0$, referred to reference systems related in different ways to G. *J. G. Semple* (London).

Hartley, E. M. A sextic primal in five dimensions. Proc. Cambridge Philos. Soc. **46**, 91–105 (1950).

This paper follows naturally from the paper by Todd which was the subject of the preceding review. Its object is to investigate in detail the properties of the invariant sextic primal Ψ of the collineation group G of S_5, using the equation for Ψ obtained by Todd. The results, on account of the complication of the configurations involved, are difficult to summarize adequately. The main fact which emerges, however, is that there exists on Ψ a remarkable set of 34,020 lines which are specially related to the configuration of the 126 vertices of homologies of G. The study and enumeration of these lines, and the fitting of them into the general configuration, provide most of the results of the paper. *J. G. Semple* (London).

Roussopoulos, A. Les chaînes dans la théorie des groupes. Bull. Soc. Math. Grèce **24**, 1–12 (1949). (French. Greek summary)

A chain is a two-way sequence of elements a_k such that $a_k a_{k+1} = a_{k+2}$ (in the group of integers, a Fibonacci series). The chains generated by ordered pairs of elements are studied analogously to the cyclic subgroups generated by single elements. A typical result concerns the length λ of the period of a chain in a finite group: If $\sum\lambda = m^2$, for a set of chains containing exactly m group elements, then these elements constitute a subgroup. This is illustrated by the calculation of all subgroups of a certain group of order 12. *R. C. Lyndon* (Princeton, N. J.).

Neumann, B. H. On a special class of infinite groups. Nieuw Arch. Wiskunde (2) **23**, 117–127 (1950).

The author answers the following question: can a group with a finite number of generators possess subgroups which cannot be generated by a finite number of their elements? He constructs a family of groups G_S with the required properties; S runs over a set of the power of the continuum and different indices S belong to different groups. Each G_S has 3 generators; the generating relations are explicitly given. In a footnote, the author points out that recent investigations [G. Higman, B. H. Neumann and H. Neumann, J. London Math. Soc. **24**, 247–254 (1949); these Rev. **11**, 322] have shown that every countable group is a subgroup of a 2-generator group. *F. W. Levi* (Bombay).

Krasner, Marc. Le produit complet et la théorie de la ramification: préliminaires. C. R. Acad. Sci. Paris **229**, 1103–1105 (1949).

Krasner, Marc. Le produit complet et la théorie de la ramification: extramodules; résumé de l'ancienne théorie. C. R. Acad. Sci. Paris **229**, 1287–1289 (1949).

Krasner, Marc. Le produit complet et la théorie de la ramification: résumé de l'ancienne théorie (fin); la nouvelle théorie. C. R. Acad. Sci. Paris **230**, 162–164 (1950).

In a joint note [same C. R. **227**, 806–808 (1948); these Rev. **10**, 351] L. Kaloujnine and the author defined the "complete product" of a set of groups, and announced various results on the isomorphisms of an abstract group A with subgroups of the complete product of the factors of certain types of "anti-invariant" series of A. In the present three notes the author outlines how these results may be used to formulate his "generalised ramification theory," as outlined in earlier notes [same C. R. **219**, 345–347, 433–435, 473–476, 539–541 (1944); **220**, 28–30, 761–763 (1945); **221**, 737–739 (1945); **222**, 37–40, 165–167, 363–365, 581–583, 626–628, 984–986 (1946); these Rev. **7**, 363, 364, 429, 510] in terms of the Galois group itself, instead of the Galois hypergroup. The discussion, and the results announced, are highly technical and cannot be mentioned here. *S. A. Jennings* (Vancouver, B. C.).

Sesekin, N. F. On the theory of special torsion-free groups. Doklady Akad. Nauk SSSR (N.S.) **70**, 185–188 (1950). (Russian)

Let G denote a special torsion-free group. The group G is an R-group (i.e., extraction of roots of elements is unique). If G has at least one maximal Abelian subgroup of finite rank, then G has a nontrivial center. If every Abelian subgroup of G has finite rank, then G has a finite ascending central series every factor of which is an Abelian torsion-free group of finite rank. *R. A. Good* (College Park, Md.).

Kontorovič, P. G. Groups with a basis of partition. IV. Mat. Sbornik N.S. **26(68)**, 311–320 (1950). (Russian)

[For part III see Mat. Sbornik N.S. **22(64)**, 79–100 (1948); these Rev. **9**, 493.] A paper by Baer [Duke Math. J. **3**, 68–122 (1937)] discusses the smallest partial reduction of an Abelian torsion-free group. The theory is here extended to R-groups [cf. Doklady Akad. Nauk SSSR (N.S.) **59**, 213–216 (1948); these Rev. **9**, 409]. Only very minor modifications (primarily in terminology) need be made in the statements of results in sections 2.4, 2.7, 9.3a, b, 9.8(I), 9.9, 9.10 in Baer's paper, in order to obtain the analogous generalized results of this paper. *R. A. Good.*

Gluškov, V. M. On normalizers of complete subgroups in a complete group. Doklady Akad. Nauk SSSR (N.S.) **71**, 421–424 (1950). (Russian)

In a complete group with ascending central series (but not, however, in an arbitrary complete group) the normalizer of an arbitrary complete subgroup is complete. In a torsion-free group with ascending central series the normalizer of a servant subgroup is a servant subgroup.

R. A. Good (College Park, Md.).

Kaloujnine, Leo. Une propriété des suites caractéristiques d'un p-groupe. C. R. Acad. Sci. Paris **229**, 1105–1107 (1949).

Let G be a p-group, H any characteristic subgroup of G, and let H^* be a characteristic subgroup of G contained in H and maximal in H in the sense that, whenever $H^* \subseteq K \subseteq H$, where K is a characteristic subgroup of G, then $H^* = K$ or $H = K$. Under these hypotheses the author proves that the order of H/H^* is at most p^δ, where

$$\delta = (p^d - p^{d-1})(p^d - p^{d-2}) \cdots (p^d - 1)$$

and d is the number of elements in a minimal basis of G. The proof depends upon the following lemma. Let L be an Abelian group of type $(p, p, \cdots, p)$ and Q a group of automorphisms of L having an invariant p-subgroup of index r in Q; then if L is Q-simple, the order of L is at most p^r.

S. A. Jennings (Vancouver, B. C.).

Meier-Wunderli, Heinrich. Über endliche p-Gruppen, deren Elemente der Gleichung $x^p = 1$ genügen. Comment. Math. Helv. **24**, 18–45 (1950).

The author shows how to choose, in a group of order p^r satisfying $x^p = 1$, elements $P_1, P_2, \cdots, P_r$ such that the defining relations between them may be taken to be $P_i^p = 1$ and

$$P_j P_i = P_i P_j \prod P_k^{\alpha^k_{ij}}, \qquad i < j,\ k = j+1, \cdots, r.$$

If the group can be generated by n elements we may further suppose that, for $k > n$, one of these relations is $P_j P_i = P_i P_j P_k$. Conversely, if we are given integers α^k_{ij}, this group will satisfy $x^p = 1$ provided the α^k_{ij} satisfy certain conditions not explicitly formulated; and will be of order p^r provided they satisfy further conditions, which ensure that for $i = 1, \cdots, r$, $\{P_i, \cdots, P_r\}$ is a cyclic extension of order p of $\{P_{i+1}, \cdots, P_r\}$.

The author states that these methods suffice for an inductive construction of $B_{n,p,c}$, the maximal n-generator group of class c satisfying $x^p = 1$. He constructs for every $p \geqq 3$ a metabelian 2-generator group of order exponent $\frac{1}{2}p(p-1) - p + 3$ satisfying $x^p = 1$, and shows that subject to these conditions it is maximal. However, for $p > 3$ he exhibits a 2-generator group of order exponent $\frac{1}{2}p(p-1)$ satisfying $x^p = 1$. Lastly, he constructs the group $B_{2,4,6}$ and shows that its order is 5^{14}, and he ascribes to P. Hall [unpublished] the result that $B_{2,4,7}$ is of order 5^{18}.

G. Higman (Manchester).

Jaffard, Paul. Théorie des filets dans les groupes réticulés. C. R. Acad. Sci. Paris **230**, 1024–1025 (1950).

Let G denote a lattice-ordered Abelian group, G^+ the set of its positive elements (including 0). The equivalence relation $a = b$ if and only if $a \cap x = 0 \leftrightarrow b \cap x = 0$ for all $x \in G^+$ divides G^+ into equivalence classes called filets; $\bar{a}$ is the filet containing a; $\bar{a} \geqq \bar{b}$ if and only if $a \cap x = 0$ implies $b \cap x = 0$ $(x \in G^+)$. Then the set of filets forms a lattice with $\bar{a} \cap \bar{b} = \overline{a \cap b}$ and $\bar{a} \cup \bar{b} = \overline{a \cup b}$. The subgroup F^* generated by a minimal filet F is linearly ordered. A subgroup generated by a set of minimal filets F_i is the direct sum of the F_i^*. No references are given. *P. M. Whitman.*

Jaffard, Paul. Applications de la théorie des filets. C. R. Acad. Sci. Paris **230**, 1125–1126 (1950).

The representation as a direct sum in the preceding review is unique. A lattice-ordered group G is said to have linear dimension n if it is the product of n linearly ordered groups. Theorem: This is true if and only if (1) given filets $\bar{a}, \bar{b}$ $(\neq 0)$, there exists $x \in \bar{a}$ such that $x \leqq b$ fails, and (2) n is the maximal order of a subset $M \subset G$ such that $N < M$ implies $\inf_{x \in N} a_i > \inf_{x \in M} a_i$. *P. M. Whitman.*

Harish-Chandra. On faithful representations of Lie groups. Proc. Amer. Math. Soc. **1**, 205–210 (1950).

Let G, H be connected Lie groups and ϕ a continuous homomorphism of H into the group of automorphisms of G. The semidirect product $G \times_\phi H$ is defined to consist of pairs (g, h) with composition $(g_1, h_1)(g_2, h_2) = (g_1(\phi(h_1)g_2), h_1h_2)$ and with the topology of the Cartesian product $G \times H$. The author proves that if G is simply connected and solvable and H has a faithful representation then $G \times_\phi H$ has a faithful representation. A theorem of Malcev [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 87–89 (1943); these Rev. **7**, 115], giving a necessary and sufficient condition that a connected solvable Lie group admit a faithful representation, is a corollary. *P. A. Smith* (New York, N. Y.).

Wever, Franz. Operatoren in Lieschen Ringen. J. Reine Angew. Math. **187**, 44–55 (1949).

Using the reviewer's notation [Substitutional Analysis, Edinburgh, 1948; these Rev. **10**, 280], the author applies Young's methods to the results concerning relations between groups and certain associative rings and Lie rings [see Magnus, J. Reine Angew. Math. **182**, 142–149 (1940); these Rev. **2**, 214]. It is shown that any left-normalised product of dimension m of the module $\mathfrak{L}$ (of the free associative noncommutative ring $\mathfrak{A}$), which represents the free Lie ring Λ, may be expressed in the form

$$[[[\cdots[[xy]z]\cdots]s]t] = \omega_m\{xyz\cdots st\},$$

where $\omega_m = (\epsilon - \pi_m)(\epsilon - \pi_{m-1})\cdots(\epsilon - \pi_2)$, ϵ is the identity

permutation, and π_k is the permutation $(12\cdots k)$. The right ideal (ω_m) of the group algebra of $\mathfrak{P}_m$ therefore furnishes a basis of all left normalised forms of dimension m in Λ_m. This basis is portrayed by a "basis operator" B_m of the group algebra when the variables are distinct, or by the operator $B_mP_1^\beta$ when the repetitions of the variables are specified by the partition β, P_1^β being a Young symmetry operator. In an analogous manner any product in Λ_m with bracketing ρ is represented by an operator $\omega_m K^\rho$. In fact the properties of ω_m determine the structure of Λ_m and it is shown that the number of irreducible representation modules associated with the partition α, in Λ_m is $(1/m)\sum_{d|m}\mu(d)\chi^\alpha(d,\cdots,d)$, $\mu(d)$ being the sum of the coefficients of the elements of the class $(d,\cdots,d)$ in ω_m.

As shown by Magnus, a free group $\mathfrak{F}$ can be associated with an associative ring $\mathfrak{A}$ in such a way that Λ_m corresponds to the Abelian factor group $\mathfrak{F}_m/\mathfrak{F}_{m+1}$. If $\mathfrak{D}^n$ be the commutator of $\mathfrak{D}^{n-1}$, $\mathfrak{D}^0=\mathfrak{F}$, the factor group $\mathfrak{D}^n\cdot\mathfrak{F}_{2^n+1}/\mathfrak{F}_{2^n+1}$ determines a submodule Δ^n in Λ_{2^n}. It is shown that the number of linearly independent forms in Δ^n with distinct generators is $(2^n)!/2^{2^n-1}$. Finally a criterion is established, which determines whether a given group can be generated by a given number of generators. *D. E. Rutherford.*

Mackey, George W. Imprimitivité pour les représentations des groupes localement compacts. II. Nombres d'entrelacement pour les représentations imprimitives. C. R. Acad. Sci. Paris **230**, 808–809 (1950).

L'auteur annonce des résultats qui font suite à ceux d'une note précédente [Proc. Nat. Acad. Sci. U. S. A. **35**, 537–545 (1949); ces Rev. **11**, 158]. Etant données deux représentations unitaires $x\to U_x$ et $x\to V_x$ d'un groupe localement compact séparable G, on appelle opérateur d'entrelacement de ces deux représentations toute application linéaire continue T de l'espace de U dans l'espace de V, telle que $V_xT=TU_x$ pour tout x; ces opérateurs forment un espace vectoriel dont la dimension ($\leqq+\infty$) est notée $I(U, V)$. Le résultat essentiel est le suivant. Soient G_1 et G_2 des sous-groupes à la fois ouverts et fermés de G, et soient L et M deux représentations unitaires de ceux-ci; soit $D=G_1xG_2$ une classe bilatère modulo G_1 et G_2; on peut alors définir deux représentations du sous-groupe $x^{-1}G_1x\cap G_2$ par les formules $s\to L_{xsx^{-1}}$ et $s\to M_s$; le nombre d'entrelacement de ces deux représentations ne dépend que de la classe D; soit $I(D, L, M)$ ce nombre; par ailleurs, soient U^L et U^M les représentations imprimitives de G engendrées par L et M; enfin, désignons par $\mathfrak{D}$ l'ensemble de toutes les classes bilatères, et par $\mathfrak{D}'$ l'ensemble des classes G_1xG_2 pour lesquelles le groupe $x^{-1}G_1x\cap G_2$ est d'indice fini dans $x^{-1}G_1x$ et dans G_2; ceci étant, on a

$$\sum_{D\in\mathfrak{D}'} I(D, L, M)\leqq I(U^L, U^M)\leqq \sum_{D\in\mathfrak{D}} I(D, L, M),$$

et si L et M sont de dimension un, la première de ces inégalités est une égalité. L'auteur tire de là des résultats qui, dans les groupes finis, sont dûs à Frobenius et à Shoda, et qui permettent de trouver des critères précis d'irréductibilité ou d'équivalence unitaire pour les représentations imprimitives des groupes discrets (entre autres).

R. Godement (Nancy).

Vilenkin, N. Ya. On the theory of general noncommutative topological groups. Doklady Akad. Nauk SSSR (N.S.) **71**, 1013–1015 (1950). (Russian)

A "general topological group" is a topological group in the usual sense except that it is not required to satisfy a separation axiom. In this note the author proves that every general topological group is the quotient of a Hausdorff topological group by a (not necessarily closed) normal subgroup. This result generalizes work in an earlier paper [same Doklady (N.S.) **58**, 1573–1575 (1947); these Rev. **9**, 326] in which the same theorem was proved for general Abelian topological groups. *G. W. Mackey* (Cambridge, Mass.).

NUMBER THEORY

Hasse, Helmut. Zahlentheorie. Akademie-Verlag, Berlin, 1949. xii+468 pp. 44 DM; bound, 47 DM.

This book is an introduction to number theory (especially algebraic number theory) from the point of view of modern algebra. The first part, covering about one-sixth of the book, develops the arithmetic of the rational number field, including the theory of quadratic residues and their reciprocity law; at the end of each section is a supplement giving the modifications necessary if one goes over to the somewhat similar case of the field of rational functions in one unknown over a finite coefficient field. The second part, forming about one-third of the book, is a detailed introduction to valuation theory in general and, specifically, to the structure theory of fields complete [perfekt] with respect to a discrete valuation such that the residue class field is finite. The third part, constituting about one-half the book, builds up the arithmetic of algebraic number fields by means of the local theory developed in part II; supplements to the various sections give the parallel arithmetic theory for algebraic function fields in one unknown over a finite field of constants, going as far as the Riemann-Roch theorem. Whenever convenient the author uses more general assumptions than those mentioned; however, he usually strikes a good balance between the concrete and the abstract. The only preparation needed for reading this work is a knowledge of the basic material on groups, rings, integral domains, fields, linear algebra, and field extensions, such as is abundantly found, for example, in the first seven chapters of van der Waerden's Moderne Algebra, v. 1, 2d ed. [Springer, Berlin, 1937]. In particular, no previous knowledge of valuation theory is assumed. Although the author has refrained from entitling his book "Zahlentheorie I," he says that he expects to publish subsequent volumes dealing with class field theory, the general reciprocity law, arithmetic of algebras, etc., so that he will have finally a comprehensive treatise on what he calls "moderne Zahlentheorie."

The titles of the sections follow. In part I: Factorization into prime numbers; Divisibility; Congruences; Structure of the residue class ring modulo m and prime residue class group modulo m; Quadratic residues. In part II: The basic concepts concerning valuations; The arithmetic in a field with discrete valuation; The completion [perfekte Hülle] of a field with valuation; The completion of a field with discrete valuation; The types of fields complete with respect to a discrete valuation such that the residue class field is perfect [vollkommen]; Continuation of a discrete valuation onto a purely transcendental extension; Continuation of the valuation of a complete field onto a finite algebraic extension; The types of fields complete with respect to an Archimedean valuation; The structure of a finite algebraic

extension of a field complete with respect to a discrete valuation; The structure of the multiplicative group of a field complete with respect to a discrete valuation such that the residue class field is perfect and has nonzero characteristic p; The types of extensions with ramification order not divisible by p of a field complete with respect to a discrete valuation such that the residue class field is finite and has characteristic p; Exponential function, logarithm, and power function in a field of characteristic 0 complete with respect to a non-Archimedean valuation; Continuation of the valuation of a noncomplete field onto a finite algebraic extension. In part III: Relations between the full system of valuations and arithmetic in the rational number field; Continuation of the full system of valuations onto a finite algebraic extension; The prime spots of an algebraic number field and their completions; Factorization into prime divisors, integrality, divisibility; Congruences; The multiples of a divisor; Different and discriminant; Quadratic number fields; Cyclotomic fields; Units; Class number; Approximation theorems and estimates of discriminants.

On the whole this work impressed the reviewer as a very sound, well-written, and important book. However, the following comments seem called for. (1) Historical remarks and references to the literature are very sparse. (2) The author eschews the "Satz-Beweis" style and prefers a freer form of presentation. However, the reviewer felt that, in certain parts of the book at least, the "Satz-Beweis" style would have made it easier for the reader to keep the logical framework in mind. (3) The author's thoroughness and attention to detail are at times a little overwhelming, although the book is actually easy to read in the small. Thus the book is (by the author's own confession) not very suitable for a textbook, since it takes so long to arrive at any of the standard theorems of algebraic number theory. On the other hand the manner of presentation referred to above and the tendency to make each section depend in detail on so much of what has gone before, make the book difficult to use as a reference book unless the user has first read it through carefully. However, such a careful reading is well worth the effort and gives one an excellent idea of recent methods (specifically, valuation theory) in algebraic number theory. *P. T. Bateman* (Urbana, Ill.).

Palamà, G. Numero di termini minimo di un membro di multigrade non banali. Boll. Un. Mat. Ital. (3) **4**, 310–317 (1949).

The author proves the following two theorems and gives some consequences of them. (1) A nontrivial multigrade equation

$$a_1, \cdots, a_{n+m} \overset{u}{=} b_1, \cdots, b_{n-1}$$

($m \geqq 1$) is impossible if all the terms are positive. (2) A nontrivial solution in integers,

$$a_1, \cdots, a_{n+m} \overset{u}{=} b_1, \cdots, b_{n-2}$$

($m \geqq 1$) is impossible if some of the a's and b's are positive and some negative. *W. H. Simons* (Kingston, Ont.).

Jacobsthal, Ernst. Zur Theorie der Bernoullischen Zahlen. Norske Vid. Selsk. Forh., Trondheim **22**, no. 24, 107–112 (1950).

Generalizing some results of Ljunggren [Avh. Norske Vid. Akad. Oslo. I. **1947**, no. 5 (1948); these Rev. **9**, 568] the following formula is proved:

$$\sum_{a=1,\,(a,m)=1}^{m} (a+mB)^{2q} = mB^{2q} \prod_{p \mid m} (1-p^{2q-1}).$$

There is a similar formula in which a is restricted to the range $1 \leqq a \leqq \frac{1}{2}(m-1)$, $m \geqq 3$. The proof makes use of a formula of Blissard [Quart. J. Pure Appl. Math. **4**, 279–305 (1861)]. *L. Carlitz* (Durham, N. C.).

Jacobsthal, Ernst. Über einige Kongruenzen. I. Norske Vid. Selsk. Forh., Trondheim **22**, no. 15, 56–61 (1950).

Let $n = p^\beta n'$, $(n', p) = 1$, $\beta \geqq 1$, $S_r(n, p) = \sum^n{}_{\lambda=1} \lambda^{-r}$, $(\lambda, p) = 1$. In connection with a paper of V. Brun [Norsk Mat. Tidsskr. **23**, 41–53 (1941), p. 52; these Rev. **7**, 106] the residue of S_r mod p^s for large s is sought. In the present note some preliminary formulas are set up. For example, if $S_r(x_1, \cdots, x_n) = \sum^n{}_{i=1} x_i^{-r}$ then

$$S_r(x_1+h, \cdots, x_n+h) \equiv \sum_{\lambda=0}^{s-1} (-1)^\lambda \binom{r+\lambda-1}{\lambda} S_{\lambda+r}(x_1, \cdots, x_n) h^\lambda$$

(mod d^s), where $d \mid h$, the x_i are prime to d, and r and s are arbitrary positive integers. *L. Carlitz* (Durham, N. C.).

Jacobsthal, Ernst. Über einige Kongruenzen. II. Norske Vid. Selsk. Forh., Trondheim **22**, no. 22, 96–101 (1950).

In the formulas of part I [see the preceding review] let the x_i be distinct and symmetric with respect to the arithmetic mean $M = n^{-1}(x_1 + \cdots + x_n)$. In particular, suppose the x_i denote those numbers of the sequence $1, 2, \cdots, N$ which are prime to d, where d is some fixed divisor of N, and let $f_r = f_r(N, d)$ be the rth elementary symmetric function of the x_i. Then it is proved that f_r is divisible by N for odd r and indeed $f_r - \frac{1}{2}(n-r+1)Nf_{r-1} \equiv 0$ (mod N^2) (r odd). A number of similar results are also proved.

L. Carlitz (Durham, N. C.).

Jacobsthal, Ernst. Über einige Kongruenzen. III. Norske Vid. Selsk. Forh., Trondheim **22**, no. 23, 102–106 (1950).

Using the notation of the two preceding reviews the following results are obtained: (1) let

$$S_r = S_r(N, d) = S_r(x_1, \cdots, x_n) = s_r(x_1^{-1}, \cdots, x_n^{-1}),$$

r odd; then $S_r \equiv 0$ (mod d); $2S_r + rNS_{r+1} \equiv 0$ (mod d^2); (2) $S_r + \frac{1}{2}rNs_{r+1} \equiv 0$ (mod d^2); (3) $(r+2)S_r + rs_{r+2} \equiv 0$ (mod d^2). Finally the author corrects a congruence of Frobenius [S.-B. Preuss. Akad. Wiss. **1910**, 809–847, in particular, p. 817, formula 5]. The correct statement is

$$S_n + (n-1)f_n \equiv 0 \pmod{p^2}, \qquad n = 3, 5, \cdots, p-2; \; p \geqq 5.$$

L. Carlitz (Durham, N. C.).

Mirsky, L. Generalization of some results of Evelyn-Linfoot and Page. Nieuw Arch. Wiskunde (2) **23**, 111–116 (1950).

Let $n, q, k, s, r_1, \cdots, r_s$ be integers such that $n \geqq 1$, $1 \leqq k \leqq q$, $s \geqq 2$, $2 \leqq r_1 \leqq \cdots \leqq r_s$, and let $Q(n)$ denote the number of representations of n in the form $n = n_1 + \cdots + n_s$, where the $n_1, \cdots, n_s$ are r_1-free, $\cdots$, r_s-free respectively, and $n_1 \equiv \cdots \equiv n_s \equiv k$ (mod q). An asymptotic formula, for $n \to \infty$, is obtained for $Q(n)$. Also conditions are given under which $Q(n) = 0$. *W. H. Simons* (Kingston, Ont.).

Rényi, Alfred. On the large sieve of Ju V. Linnik. Compositio Math. **8**, 68–75 (1950).

The results of this paper are essentially those reviewed previously [J. Math. Pures Appl. (9) **28**, 137–149 (1949); these Rev. **11**, 161]. The only difference is that in the former paper the "large sieve" is derived from a general probabilistic theorem whereas in the present paper a direct and self-contained approach is used. *M. Kac.*

Lehner, Joseph. Proof of Ramanujan's partition congruence for the modulus 11^3. Proc. Amer. Math. Soc. 1, 172–181 (1950).

The author [Amer. J. Math. 65, 492–520 (1943); these Rev. 5, 34] has verified Ramanujan's conjecture $p(11^{\alpha}l+\rho)\equiv 0 \pmod{11^{\alpha}}$ if $24\rho\equiv 1 \pmod{11^{\alpha}}$, for $\alpha=1, 2$. The proof is now extended to include the case $\alpha=3$ by making use of the inductive formulas for $L(r, 11^{\alpha})$, the modular properties of certain associated functions and the explicit values of a number of the coefficients in their expansions.

H. S. Zuckerman (Seattle, Wash.).

Brigham, Nelson A. A general asymptotic formula for partition functions. Proc. Amer. Math. Soc. 1, 182–191 (1950).

Let $\gamma(n)$ be a nonnegative function and let $g_{\gamma}(s)$, $a_{\gamma}(k)$ be defined by the relation

$$g_{\gamma}(s)=\prod_{n=1}^{\infty}(1-e^{-ns})^{-\gamma(n)}=\sum_{k=0}^{\infty}a_{\gamma}(k)e^{-ks} \qquad (\Re s>0).$$

Then $a_{\gamma}(k)$ is a weighted partition function of k, given by

$$a_{\gamma}(k)=\sum^{l}\prod_{n=1}\gamma(n)[\gamma(n)+1]\cdots[\gamma(n)+k_n-1]/k_n!,$$

where the sum is extended over all partitions of k having the form $k=k_1+2k_2+\cdots+lk_l$ $(k_1, \cdots, k_l>0)$. Using a method due to G. H. Hardy and S. Ramanujan [Proc. London Math. Soc. (2) 16, 112–132 (1917)] the author proves that if $\sum_{n\leq x}\gamma(n)\sim Kx^u\log^v x$ $(x\to\infty; K>0, u>0)$, then the summatory function $A_{\gamma}(k)=\sum_{n=0}^{k}a_{\gamma}(n)$ satisfies, for $k\to\infty$, the relation

$$(*)\quad \log A_{\gamma}(k)\sim u^{-1}\{Ku\Gamma(u+2)\zeta(u+1)\}^{1/(u+1)}\times(u+1)^{(v-v)/(u+1)}k^{u/(u+1)}(\log k)^{v/(u+1)}.$$

This result is established by showing that

$$\log g_{\gamma}(s)\sim K\Gamma(u+1)\zeta(u+1)s^{-u}(\log s^{-1})^{v}$$

$(s\to+0)$ and then appealing to a Tauberian theorem on general Dirichlet series with positive coefficients [Hardy and Ramanujan, loc. cit., theorem A].

The author also shows that under certain conditions the function $A_{\gamma}(k)$ in (*) may be replaced by $a_{\gamma}(k)$. Finally, he discusses a number of applications of (*) and points out, in particular, that by appropriate specializations of $\gamma(n)$ we can obtain the Hardy-Ramanujan formulae involving unrestricted partitions, partitions into rth powers, and partitions into primes. *L. Mirsky* (Sheffield).

Brigham, Nelson A. On a certain weighted partition function. Proc. Amer. Math. Soc. 1, 192–204 (1950).

The author observes that no asymptotic formula for the number of partitions of a large integer into primes has as yet been found. As a contribution to this problem he obtains, on the assumption of the Riemann hypothesis, an asumptotic formula for $\sum_{n=0}^{k}a(n)$, where $a(k)$ is a weighted partition function involving primes and powers of primes, which is defined by the expansion

$$g(s)=\prod_{n=1}^{\infty}(1-e^{-ns})^{-\Lambda(n)}=\sum_{k=0}^{\infty}a(k)e^{-ks},$$

where $\Lambda(n)$ has its customary meaning. [For the precise arithmetical significance of $a(k)$ see the preceding review.]

The first step in the argument is to apply Mellin's inversion formula; this enables us to show that, as $s\to+0$ in the region $|\arg s|\leqq\theta_0<\frac{1}{2}\pi$,

$$(*)\qquad g(s)=K_2s^{\frac{1}{2}\alpha+2\nu}\exp\{\tfrac{1}{8}\pi^2/s+Z(s)+O(|s|)\};$$

here

$$K_2=\exp(-1+\pi^2/24+\textstyle\sum_{\rho}\rho^{-2}),$$
$$Z(s)=\textstyle\sum_{\rho}s^{-\rho}\zeta(1+\rho)\Gamma(\rho),$$

and ρ ranges over the complex zeros of $\zeta(s)$, multiple ones being counted according to their multiplicity. A Tauberian theorem due to A. E. Ingham [Ann. of Math. (2) 42, 1075–1090 (1941), theorem 1; these Rev. 3, 166] is now made use of; to verify that all its conditions are satisfied it is necessary to assume the full strength of the Riemann hypothesis. Ingham's theorem and relation (*) lead to the conclusion that, for $k\to\infty$,

$$\sum_{n=0}^{k}a(k)\sim K_1\sigma^{\frac{1}{2}+\frac{1}{2}\alpha+2\nu}\exp\{\sigma^{-1}(\tfrac{1}{8}\pi^2+\sigma Z(\sigma)-\sigma^2Z'(\sigma))\},$$

where $K_1=(3/2\pi)^{\frac{1}{2}}K_2/\pi$ and σ is the unique positive solution of the equation $\pi^2/6\sigma^3-Z'(\sigma)=k$. *L. Mirsky* (Sheffield).

Skolem, Th. Remarks on the representation of natural numbers as sums of three or four squares. Norske Vid. Selsk. Forh., Trondheim 21, no. 39, 172–175 (1949).

New proofs are given of the following two theorems, the first being used to prove the second. If the positive integer m is a divisor of the norm $N(A)$ of a Lipschitz integral quaternion A, then there exists an integral quaternion B of norm m such that m is a divisor of $A'B$, where A' denotes the conjugate of A. (Proof is given for m odd.) If an integer m is a sum of three rational squares, it is a sum of three integral squares. The proof of the first theorem is based on previous work of the author [Norsk Mat. Tidsskr. 25, 76–87 (1943); these Rev. 8, 7] on the arithmetic of quaternions.

I. Niven (Eugene, Ore.).

Skolem, Th. Proof of a theorem on 3-lattices. Norske Vid. Selsk. Forh., Trondheim 21, no. 44, 197–200 (1949).

Another proof is given of the theorem that in a 3-dimensional lattice there is always a fundamental parallelepiped having the product of its sides not exceeding $\sqrt{2}$ times its volume. Let $\overline{OX}$ be the length of the vector OX. The proof uses only the inequalities that if OA, OB, OC generate a parallelepiped for which $\overline{OA}\cdot\overline{OB}\cdot\overline{OC}$ is minimal then $\overline{OA\pm OB}\geqq\overline{OB}$ while $\overline{OA\pm OC}$, $\overline{OB\pm OC}$, and $\overline{OA\pm OB\pm OC}$ all are not less than $\overline{OC}$. *L. Tornheim.*

Davenport, H. Indefinite binary quadratic forms. Quart. J. Math., Oxford Ser. (2) 1, 54–62 (1950).

Let $f(x, y)=ax^2+bxy+cy^2$ have real coefficients and discriminant $d=b^2-4ac>0$, and $f(x, y)\neq 0$ for integral $(x, y)\neq(0, 0)$. Then there exist real numbers ξ, η such that $|f(x+\xi, y+\eta)|>\kappa^2\sqrt{d}$ for all integral x, y, where κ is an absolute constant. If a, b, c are integers then ξ, η can be taken to be rational. In the proof ξ, η are given explicitly, based upon a "regular" chain of equivalent forms selected from a complete chain of reduced forms according to Hermite. The author states that he has shown in a paper in the course of publication that κ^2 may be taken as $1/128$, but the present proof is based on more general ideas, and the method can be extended to cubic fields of negative discriminant. *L. Tornheim* (Ann Arbor, Mich.).

Mordell, L. J. The minimum of a binary cubic form. Acta Univ. Szeged. Sect. Sci. Math. 13, 69–76 (1949).

Lecture at the University of Szeged.

Mahler, K. On Dyson's improvement of the Thue-Siegel theorem. Nederl. Akad. Wetensch., Proc. 52, 1175–1184 = Indagationes Math. 11, 449–458 (1949).

F. J. Dyson verschärfte [Acta Math. 79, 225–240 (1947); diese Rev. 9, 412] den Thue-Siegelschen Satz, indem er zeigte: Wenn zu irgendeiner algebraischen Zahl ξ vom Grade $n \geqq 2$ bei positivem μ die Ungleichung $|\xi - p/q| < q^{-\mu}$ mit ganzen Zahlen p, q und $q \geqq 1$ unendlich viele solcher Lösungen p/q besitzt, so ist $\mu \leqq (2n)^{\frac{1}{2}}$. In dieser Arbeit wird ein vereinfachter, besonders durchsichtiger Beweis des Haupthilfssatzes der Dysonschen Arbeit gegeben. Am Schluss wird ein weiterer ähnlicher Hilfssatz mit Anwendung auf Kettenbrüche algebraischer Zahlen angekündigt. [Ein Beweis des Referenten für den genannten Dysonschen Satz ist fälschlich mit Mathematische Nachrichten 2, 288–295 (1949), statt wie es richtig heissen muss Archiv der Mathematik 1, 288–295 (1949) zitiert; vgl. diese Rev. 10, 592.]

T. Schneider (Göttingen).

Jarník, Vojtěch. Une remarque sur les approximations diophantiennes linéaires. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 82–86 (1950).

Let $S = \{\theta_1, \theta_2, \cdots, \theta_r\}$ be a system of r real numbers, and let $t \geqq 1$. Put $\psi_s(t) = \min |a_1\theta_1 + \cdots + a_r\theta_r + a_0|$ where the minimum extends over all integers a_i for which

$$0 < \max(|a_1|, \cdots, |a_r|) \leqq t.$$

Assume $\psi_s(t) > 0$ for all $t \geqq 1$, so that $1, \theta_1, \theta_2, \cdots, \theta_r$ are linearly independent over the rational field. The author proves the following results. (1) If $A = \liminf_{t\to\infty} t^r\psi_s(t) = A > 0$, $B = \limsup_{t\to\infty} t^r\psi_s(t)$, then $(B/A)^{r} \geqq 2$. (2) If $r = 2$, then $A \leqq 36B^3$, which is better than (1) when B is small. The inequality (2) is an immediate consequence of the following result. (3) Let $r = 2$. Denote by $\varphi(t)$ a continuous steadily decreasing function of $t \geqq 1$ such that $t\varphi(t) \to 0$ as $t \to \infty$. If $\psi_s(t) < \varphi(t)$ for all sufficiently large t, then there exist arbitrarily large t such that $\psi_s(t) < \varphi(1/6t\varphi(t))$.

K. Mahler (Manchester).

ANALYSIS

Obrechkoff, Nikola. Sur quelques propriétés des dérivées des fonctions d'une variable réelle. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 231–235 (1950).

Let $f(x)$ be a real function with an nth derivative which is nonnegative for $x > a$. Let $\lim_{\lambda\to\infty} f(y_\lambda)/y_\lambda^m = 0$, $y_\lambda \to \infty$, $0 \leqq m < n$. Then for $x > a$ we have $(-1)^k f^{(n-k)}(x) \geqq 0$ for $k = 1, \cdots, n-m$; and $\lim_{x\to\infty} f^{(i)}(x)x^{i-m} = 0$ for $0 \leqq i \leqq m-1$, $\lim_{x\to\infty} f^{(i)}(x) = 0$ for $m \leqq i \leqq n-1$. Furthermore, if $f^{(i)}(b) = 0$, $b > a$, $m \leqq i \leqq n-1$, then $f^{(i)}(x) \equiv 0$ for $x > b$. From this result the author derives several corollaries, such as the following. If $f^{(n)}(x) \geqq 0$ and $|f(x)| < K(1 + |x|^m)$ for $-\infty < x < \infty$, $f(x)$ is a polynomial. If $|f(x)| < Ke^x$ and

$$(-1)^n \sum_{\nu=1}^{n} \binom{n}{\nu} (-1)^\nu f^{(\nu)}(x) \geqq 0$$

then $f(x) = K_1 e^x$. Similar results are given for sequences.

R. P. Boas, Jr. (Evanston, Ill.).

Mikusiński, Jan G.-. Sur les zéros des polynomes et de leurs dérivées successives. Prace Mat.-Fiz. 47, 21–40 (1949).

Let $f(x)$, defined for $a \leqq x \leqq b$, have n continuous derivatives. If all the roots of the equations $f^{(i)}(x) = 0$ are simple and if r_i is the number of roots of the ith equation then by Rolle's theorem $r_{i-1} \leqq r_i + 1$. The question arises as to whether this is the only relation between the r_i. The author answers this question in the affirmative by proving that if nonnegative integers r_i, $i = 1, \cdots, n$, are given such that $r_{i-1} \leqq r_i + 1$ then there exists a polynomial $f(x)$ of degree $r_n + n$ such that $f^{(i)}(x) = 0$ has r_i simple roots in $a \leqq x \leqq b$ for $i = 1, \cdots, n$.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Hirschman, I. I., Jr. On the distributions of the zeros of functions belonging to certain quasi-analytic classes. Amer. J. Math. 72, 396–406 (1950).

An infinitely differentiable function $f(x)$, defined on the whole axis, is said to belong to the class $C\{M_n, k\}$ if $|f^{(n)}(x)| \leqq Ak^n M_n$. Only quasi-analytic classes are considered here. The author defines regular quasi-analytic classes by imposing a condition of regularity on the M_n's. Let (Zu) be the number of zeros of $f(x)$ in the interval $[-u, u]$ counted according to their multiplicities. It is shown that if $f(x)$ belongs to a regular quasi-analytic class then $Z(u)$ can be majorized by means of a function depending on the sequence $\{M_n\}$ and the constant k. Thus the author extends to regular quasi-analytic functions a known result in the case of analytic functions belonging to the class $C\{n!, k\}$. A typical result is the following. If $f(x) \in C\{n!(\log[n+e])^n, k\}$ then $\limsup (2/\pi u) \log\log Z(u) \leqq k$. The exactness of the general result obtained is shown by constructing counter-examples.

S. Agmon (Houston, Tex.).

Good, I. J. A proof of Liapounoff's inequality. Proc. Cambridge Philos. Soc. 46, 353 (1950).

With $\mu_k = \int_0^\infty x^k dF(x)$, F nondecreasing, the author gives a short proof that $\prod_{i=1}^n \mu_{a_i} - \mu_b^n \geqq 0$, where the a_i are real and b is their arithmetic mean. When there are only two different a_i's, this reduces to Liapounoff's inequality.

R. P. Boas, Jr. (Evanston, Ill.).

Chakravarti, M. C. On the ratio of the mean deviation to the standard deviation. Calcutta Statist. Assoc. Bull. 1, no. 4, 187–190 (1948).

For fixed n, let S be a set of n real numbers of sum 0 whose absolute values a_i satisfy $\sum(a_i)^2 = n$, and let $\sum a_i = n\lambda(S)$. Then $\lambda(S)$ has (i) for each n the minimum $(2/n)^{\frac{1}{2}}$, (ii) for even n the maximum 1, (iii) for odd n the maximum $(1 - n^{-2})^{\frac{1}{2}}$. [The reviewer observes that, for sets S of n complex numbers of sum 0, the minimum is the same as above, while the maximum is 1 for every n.]

L. C. Young.

Horváth, J. On a theorem of Børge Jessen. I. Norske Vid. Selsk. Forh., Trondhjem 20, no. 12, 45–47 (1948).

For finite sets $(x_1, \cdots, x_n)$, the fundamental Kolmogoroff-Nagumo theorem furnishes a characterization of mean value functions of the form

$$m_n(x_1, \cdots, x_n) \equiv \varphi^{-1}\left\{\sum_{j=1}^{n} \varphi(x_j)/n\right\}, \quad n = 1, 2, \cdots,$$

where $\varphi(x)$ is continuous and strictly increasing. For functions $f(x)$, the extension to an analogous characterization of mean value functions of the form

$$m[f(x)] \equiv \varphi^{-1}\left\{\int_0^1 \varphi[f(x)]dx\right\}$$

was given independently by Jessen and de Finetti. The author now extends the above characterization to means of functions of several variables, establishing the following result. Theorem A. To any functional operator M defined on all measurable functions $f(x_1, \cdots, x_n)$, $0 \leqq x_j \leqq 1$, with $A \leqq f \leqq B$, there corresponds a continuous and strictly increasing function $\varphi(x)$ such that

$$M[f(x_1, \cdots, x_n)] \equiv \varphi^{-1}\left\{\int_0^1 \cdots \int_0^1 \varphi[f(x_1, \cdots, x_n)]dx_1 \cdots dx_n\right\}$$

if and only if M is symmetric, consistent, strictly increasing, associative, and (this condition is here redundant) continuous. *E. F. Beckenbach* (Los Angeles, Calif.).

Horváth, J. On a theorem of Børge Jessen. II. Norske Vid. Selsk. Forh., Trondhjem **20**, no. 13, 48–51 (1948).

Proof of theorem A of the preceding review. *E. F. Beckenbach* (Los Angeles, Calif.).

Fenyö, Stephan. The notion of mean-values of functions. Norske Vid. Selsk. Forh., Trondheim **21**, no. 38, 168–171 (1949).

The result established in this paper is described in the following review; it is an extension, to means $m(F)$ of functions, of a result of Aczél [same Forh. **19**, no. 23, 83–86 (1947); Bull. Amer. Math. Soc. **54**, 392–400 (1948); these Rev. **8**, 504; **9**, 501] concerning means of a fixed finite number of values. The extended result is obtained by applying the known result to a step function approximating a nonincreasing rearrangement of F. *E. F. Beckenbach.*

Aczél, John. New proof and extension of St. Fenyö's theorem on mean values of functions. Norske Vid. Selsk. Forh., Trondhjem **22**, no. 1, 1–4 (1949).

Fenyö has shown [see the preceding review] that in the Jessen–de Finetti extension of the Kolmogoroff–Nagumo theorem [see the second and third preceding reviews] the conditions of symmetry and associativity can be replaced by a condition of two-dimensional symmetry: $m_y m_x[F(x, y)] = m_y m_x[G(x, y)]$ whenever $F(x, y) \sim G(x, y)$ are equimeasurable, where $m_x[F(x, y)]$ denotes the mean relative to x with y constant. This condition of two-dimensional symmetry can be replaced by the two conditions of simple symmetry: $m[f(x)] = m[g(x)]$ whenever $f(x) \sim g(x)$, and reflectibility: $m_y m_x[F(x, y)] = m_y m_x[H(x, y)]$, where $H(x, y) = F(x, y)$ except on two arbitrary rectangles $a \leqq x \leqq b$, $c \leqq y \leqq d$ and $c \leqq x \leqq d$, $a \leqq y \leqq b$, on which $H(x, y) = F(y, x)$. The two conditions together are slightly weaker than two-dimensional symmetry.

It is now shown that if the condition of two-dimensional symmetry is replaced by the condition of reflectibility alone, without the condition of simple symmetry, then there results a characterization of functions of the form

$$m[f(x)] = \varphi^{-1}\left\{\int_0^1 \varphi[f(x)]dw(x)\right\},$$

where $\varphi(x)$ is continuous and strictly increasing, and $w(x)$ is an increasing weight function. If now the condition of consistency ($m(c) = c$ for constant functions $f(x) \equiv c$) also is dropped, there results a further characterization: a functional operator $m[f(x)]$ admits a representation of the form

$$m[f(x)] = \varphi^{-1}\left\{\int_0^1 \varphi[f(x)]dw(x) + K\right\},$$

where K is a constant, if and only if $m(x)$ is continuous, strictly increasing, and reflectible. *E. F. Beckenbach.*

Aczél, Jean, Fenyö, István, et Horváth, Jean. Sur certaines classes de fonctionnelles. Portugaliae Math. **8**, 1–11 (1949).

The results given in the preceding review are somewhat extended and simplified. *E. F. Beckenbach.*

Calculus

***Toeplitz, Otto. Die Entwicklung der Infinitesimalrechnung. Eine Einleitung in die Infinitesimalrechnung nach der genetischen Methode. Erster Band.** Aus dem Nachlass herausgegeben von Dr. Gottfried Köthe. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LVI. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1949. ix+181 pp. DM 19.60; bound, DM 22.60.

This is meant to be an introductory textbook in calculus in which the leading ideas and concepts are presented as they have gone through various stages of development from about 600 B.C. onward. It is a posthumous work and uncompleted, but by no means fragmentary and it contains good chapters on limits, on numbers and decimal expansions, on areas, volumes and definite integrals, on indefinite integrals and differentiation, on logarithms, on trigonometric functions, on falling bodies and planetary motion, etc. It is not meant to be a history of the topic as such, and it perhaps also falls short of being one, but it contains valuable historical remarks and, among other things, it manages to settle very nicely the controversy about the distribution of priority as among Barrow, Newton, Leibnitz and their partisans. *S. Bochner* (Princeton, N. J.).

QA303.L745 (1st, 1934)

***Lindelöf, Ernst. Einführung in die höhere Analysis.** Nach der ersten schwedischen und zweiten finnischen Auflage deutsch herausgegeben von Egon Ullrich. 2d ed. B. G. Teubner, Leipzig, 1950. ix+526 pp. $3.55.

This is an unchanged reprint of the 1st edition [1934] of this well-known work. Contents: Die elementaren Funktionen; Das Rechnen mit Näherungswerten; Kettenbrüche; Über Grenzwerte; Ableitungen von Funktionen; Länge, Flächeninhalt und Volumen; Integrale und ihre Anwendungen; Der reelle Zahlbereich; Der komplexe Zahlbereich; Die Elemente der Lehre von den linearen Gleichungen und den Determinanten; Anwendung der Determinanten zur Inhaltsberechnung für Polygone und Polyeder.

Gammel, J. L. A differentiation formula. Amer. Math. Monthly **57**, 96–99 (1950).

This note gives a proof of the following theorem. If $F(x^1, \cdots, x^N, \cdots, x^{(J)1}, \cdots, x^{(J)N})$ is either a scalar or a vector point function of class M, then along a parameterized arc C: $x^\alpha = x^\alpha(t)$ of class $M+J$, and for $0 \leqq P \leqq M$ and $0 \leqq A \leqq M+J$, the following relation holds:

$$F^{(P)}_{;(A)\alpha} = \sum_{i=0}^{J} \binom{P}{A-i} F^{(P+i-A)}_{;(i)\alpha}.$$

Here

$$F^{(P)}_{;(A)\alpha} = \partial F^{(P)}/\partial x^{(A)\alpha}, \quad x^{(A)\alpha} = d^A x^\alpha / dt^A,$$

and

$$F^{(P)} = d^P F / dt^P.$$

It is shown that the formula can be used to derive relations between the binomial coefficients. *L. A. MacColl.*

Ważewski, T. Quelques démonstrations uniformes pour tous les cas du théorème de l'Hôpital. Généralisations. Prace Mat.-Fiz. **47**, 117–128 (1949).

The main purpose of the paper is to devise a proof of L'Hospital's rule which disposes of all the special cases without the need of separate consideration of various cases. This purpose is accomplished with the aid of the following lemma. Let $\{a_\nu\}$, $\{b_\nu\}$ be sequences of real numbers such that $b_\nu \neq 0$ and either (1) $\lim a_\nu = \lim b_\nu = 0$ or (2) $\lim |b_\nu| = \infty$. Then there exist two increasing sequences of indices $\gamma_1, \gamma_2, \cdots, \delta_1, \delta_2, \cdots$ such that $b_{\delta_\nu} \neq b_{\gamma_\nu}$ and

$$\lim a_{\gamma_\nu}/b_{\delta_\nu} = \lim b_{\gamma_\nu}/b_{\delta_\nu} = 0.$$

The proof of the lemma envisages two cases. Thus, in fact, two cases of L'Hospital's rule are considered, according as in the quotient $f(x)/g(x)$ we have $\lim f(x) = \lim g(x) = 0$ or $\lim |g(x)| = \infty$. No classification based on the limit approached by x or on the behavior of $f'(x)/g'(x)$ is needed. Variations of the first proof are considered. The paper also contains a generalized form of L'Hospital's rule applicable to absolutely continuous functions. *A. E. Taylor.*

Mahajani, G. S., and Thiruvenkatachar, V. R. A note on generalised mean value theorems. Proc. Indian Acad. Sci., Sect. A. **31**, 124–129 (1950).

The principal result of the paper is the formula

$$D^n\left\{\frac{f(x)-f(a)}{x-a}\right\} \Big/ D^p\left\{\frac{\phi(x)-\phi(a)}{x-a}\right\} = \left(\frac{\xi-a}{x-a}\right)^{n-p} \frac{f^{(n+1)}(\xi)}{\phi^{(p+1)}(\xi)},$$

where $D \equiv d/dx$, and ξ is between x and a. The proof of this formula is thrown back to Cauchy's mean value theorem by the change of variable $x-a=e^t$ and suitable devices. An equivalent formula is

$$\frac{f(a)-f(x)-(a-x)f'(x)-\cdots-(a-x)^n f^{(n)}(x)/n!}{\phi(a)-\phi(x)-(a-x)\phi'(x)-\cdots-(a-x)^p\phi^{(p)}(x)/p!} = (p!/n!)(a-\xi)^{n-p} f^{(n+1)}(\xi)/\phi^{(p+1)}(\xi).$$

Various forms of the remainder in Taylor's theorem can be deduced from this. More complicated generalizations are also considered. *A. E. Taylor* (Los Angeles, Calif.).

Brand, Louis. The vector potential of a solenoidal vector. Amer. Math. Monthly **57**, 161–167 (1950).

The author considers the problem of determining the vector potential $\mathbf{g}$ of a divergence-free vector field $\mathbf{f}$ by means of simple integration. To do this, he introduces a simple closed curve C, determined by the radial vector field $\mathbf{r}(s)$ (the variable s denotes arc length along C) and constructs the cone with origin as vertex and C as directrix. By use of Stokes' and Green's formula, it is shown that $\mathbf{g}(\mathbf{r}) = -\mathbf{r}\times\int_0^1 \mathbf{f}(\lambda\mathbf{r})\lambda d\lambda$. The above formula fails to be valid if $\mathbf{r}\times\mathbf{f}(\lambda\mathbf{r}) = 0$ or if the integral is improper. In the non-exceptional cases, it is shown that the above formula, which the author states is due to Liebmann [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. **60**, 176–189 (1908)], determines the vector potential $\mathbf{g}$. That is, $\mathbf{f} = \operatorname{curl} \mathbf{g}$. Finally, the desired formulas for $\mathbf{g}$ in the exceptional cases are determined. *N. Coburn* (Ann Arbor, Mich.).

Truesdell, C. A. A form of Green's transformation. Naval Research Laboratory Rep. **3554**, iv+4 pp. (1949).

The author considers Euclidean three-space and determines a general form of Green's theorem which appears to have some uses in applied mathematics. The formula involves a tensor constructed from two arbitrary vector fields and the radius vector field. It is shown that special cases of this form of Green's theorem reduce to formulas of Burgatti, Lamb, Thomson, and Poincaré [P. Burgatti, Boll. Un. Mat. Ital. (1) **10**, 1–5 (1931); A. H. Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932, § 153; J. J. Thomson, A Treatise on the Motion of Vortex Rings, Macmillan, London, 1883, § 4; H. Poincaré, Théorie des tourbillons, Paris, 1893, § 115]. *N. Coburn.*

Theory of Sets, Theory of Functions of Real Variables

Fraïssé, Roland. Sur une nouvelle classification des systèmes de relations. C. R. Acad. Sci. Paris **230**, 1022–1024 (1950).

A polyrelation in a set E is a function A defined on a set I with values which are either elements of E or relations of several variables in E. The present note announces refinements of the classification described in an earlier note [same C. R. **228**, 1682–1684 (1949); these Rev. **11**, 17]. The signature s of A is the function defined on I whose value n_i at i is the number of variables in $A(i)$, or 0 if $A(i)$ is not a relation. We say that A is "above" B if A is similar to a restriction of B to a suitable subset of its domain. An inductive definition of (n, t)-similarity is made, where n is an ordinal and t a signature, wherein A and B are $(0, t)$-similar if they have the same signature and a common upper bound. For reasons of brevity, the rest of the definition, and the results announced in terms of it, cannot be reproduced here. *R. Arens* (Los Angeles, Calif.).

Cuesta, N. Ordering of infinitesimals. Revista Mat. Hisp.-Amer. (4) **9**, 131–140 (1949). (Spanish)

Let α and β be order types. The author writes $\alpha \leqq' \beta$ if an ordered set of type β contains a subset of type α; $\alpha =' \beta$ if $\alpha \leqq' \beta$ and $\beta \leqq' \alpha$; and $\alpha <' \beta$, if $\alpha \leqq' \beta$ but not $\beta \leqq' \alpha$. If a new element is placed between the initial part and the remainder of every decomposition of an ordered set of type τ, an ordered set of type $\varphi(\tau)$ is obtained, and $\tau <' \varphi(\tau)$. By iterating this operation, the order type $\varphi_\xi(\tau)$ is defined for any ordinal number ξ, using transfinite induction. (E.g., $\varphi_\omega(1)$ is the type of the rational numbers in their natural order.) The author uses the word "infinitesimal" to denote an ordinary null sequence of positive real numbers. By $a <'' b$ he means that nearly all the terms of the infinitesimal a are less than the respective terms of the infinitesimal b. The set S of all infinitesimals is partially ordered by the relation $<''$. Let M, with order type μ, stand for a maximal fully ordered subset of S. It is shown that $\varphi_{\omega_1}(1) \leqq' \mu$, and it follows that $|\mu| = \aleph$. An infinitesimal belongs to the subset C (D) of S if the corresponding infinite series converges (diverges). There exist M's such that both MC and MD are not empty, and $MC+MD$ is then actually a decomposition of M, whose character is not $(1, 1)$, (ω, ω^*), $(\omega, 1)$, $(1, \omega)$. With an eye to establishing some sort of absolute scale of convergence (the case of divergence is analogous), the author asks whether C contains a fully ordered subset L such that, for every $c \in C$, $c <'' x$ has a solution $x \in L$. [The second inequality sign in line 11 on p. 135 should be reversed.] *F. Bagemihl* (Rochester, N. Y.).

Iseki, Kiyosi. Simple construction of generalized Peano curve. J. Osaka Inst. Sci. Tech. Part I. **1**, 1–2 (1949).

Modifying a method by Lebesgue, the author gives the following example of a Peano curve $x_n = \phi_n(t)$ $(n = 1, 2, 3, \cdots;$

$0 \leqq t \leqq 1$) passing through all points of Hilbert space satisfying $0 \leqq x_n \leqq 1$:

$$\phi_n(t) = \sum_{k=1}^{\infty} 2^{-k} f(3^{2^{n-1}-1+2^n(k-1)}t),$$

where $f(t)$ is defined for $t \geqq 0$, continuous, and

$$f(t) = \begin{cases} 0, & 2m-\frac{1}{3} \leqq t \leqq 2m+\frac{1}{3}, \\ 1, & 2m+\frac{2}{3} \leqq t \leqq 2m-\frac{2}{3}. \end{cases}$$

F. A. Behrend (Melbourne).

Eggleston, H. G. A characteristic property of Hausdorff measure. J. London Math. Soc. **25**, 39–46 (1950).

A real-valued function $h(x)$, $0 \leqq x < +\infty$, is termed a measure function if (1) $h(x)$ is continuous and increasing, (2) $h(0)=0$, (3) $\liminf h(ax)/h(x) > 0$ for $0 < a < 1$. Now let X be a separable metric space and let E be a subset of X. For fixed $\delta > 0$, let $U(\delta, E)$ be a family of subsets of X which cover E and have diameters $d < \delta$. Take the greatest lower bound of $\sum h[d(e)]$, $e \in U(\delta, E)$, for all such coverings $U(\delta, E)$, and then take the limit of this greatest lower bound as $\delta \to 0$. This limit is h.m. E, the h measure of E. Now consider a subset A of X, a Carathéodory outer measure $F(a)$ defined for all members a of an additive class M of subsets of A, and suppose that the following conditions hold: (1) $F(A) = h.\text{m. } A < \infty$; (2) for arbitrary $\epsilon > 0$, $\eta > 0$, there exist a subset B of A and a positive number δ such that $F(A-B) < \epsilon$ and $F(B \cdot Y) \leqq (1+\eta)h[d(Y)]$ for every subset Y of X with diameter less than δ; (3) M contains all sets which are open relative to A. Then $\Phi(a) = h.\text{m. } a$ for every subset a of A which is h measurable and for which $\Phi(a)$ is defined. This uniqueness theorem is one of several results of this paper which extend previous results of Besicovitch and Moran, concerned with linear sets, to separable metric spaces. *T. Radó* (Columbus, Ohio).

Kolmogorov, A. N. On the exposition of the foundations of Lebesgue's theory of measure. Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 211–213 (1950). (Russian)

The author sketches a method whereby the original definition of measure given by Lebesgue can be applied to obtain countably additive measures on certain abstract sets. Let U be any set, and let $\mathfrak{E}$ be a family of subsets Δ of U, designated as elementary sets. Let $m(\Delta)$ be a nonnegative set-function defined for all $\Delta \in \mathfrak{E}$. Let A be any subset of U. If there exists a countable subfamily $\{\Delta_n\}$ of $\mathfrak{E}$ such that $A \subset \bigcup \Delta_n$, let the set-function $\lambda(A)$ be defined as $\inf \sum m(\Delta_n)$, the infimum being taken over all countable families $\{\Delta_n\} \subset \mathfrak{E}$ such that $A \subset \bigcup \Delta_n$. If no such family $\{\Delta_n\}$ exists, let $\lambda(A) = +\infty$. A subset B of U is said to be Lebesgue measurable if, for all $\Delta \in \mathfrak{E}$, the equality $\lambda(\Delta) = \lambda(\Delta \cap B) + \lambda(\Delta \cap B')$ obtains. The author states that λ is a Carathéodory outer measure, and that measurability in the sense of Lebesgue is equivalent to the usual measurability in the sense of Carathéodory. [Proofs of these assertions are easily supplied.] Let λ, when defined only on the family $\mathfrak{S}_\mu$ of Lebesgue measurable sets, be denoted by the symbol μ. It is next stated that a set-function μ defined on a family $\mathfrak{S}_\mu$ of subsets of U (such that $0 \leqq \mu(A) \leqq +\infty$) can be obtained by the construction described above if and only if the following conditions are satisfied: (1) the family of sets $\mathfrak{S}_\mu$ contains a largest set and is closed under the formation of countable unions and intersections; (2) μ is countably additive; (3) if $A \in \mathfrak{S}_\mu$, $\mu(A) = 0$, and $B \subset A$, then $B \in \mathfrak{S}_\mu$. Finally, the author states that, under the above construction, all elementary sets are measurable and $m(\Delta) = \mu(\Delta)$ for all elementary sets Δ if and only if the following conditions are satisfied: (4) if $\Delta \in \mathfrak{E}$ and $\{\Delta_n\}$ is a countable subfamily of $\mathfrak{E}$ such that $\Delta \subset \bigcup \Delta_n$, then $m(\Delta) \leqq \sum m(\Delta_n)$; (5) given Δ and $\Delta^* \in \mathfrak{E}$ and $\epsilon > 0$, there exist countable subfamilies $\{\bar{\Delta}_i\}$ and $\{\bar{\bar{\Delta}}_j\}$ of $\mathfrak{E}$ such that $\Delta \cap \Delta^* \subset \bigcup \bar{\Delta}_i$, $\Delta \cap \Delta^{*\prime} \subset \bigcup \bar{\bar{\Delta}}_j$, and

$$\sum m(\bar{\Delta}_i) + \sum m(\bar{\bar{\Delta}}_j) \leqq m(\Delta) + \epsilon.$$

[Reviewer's note. The construction given has the odd feature that, in some simple cases, there are no Lebesgue measurable sets. For example, let U be any infinite set, and let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be a countably infinite subset of U. Let $\mathfrak{E}$ be the family of all sets consisting of a single x_n. Let $m(x_n) = \alpha_n$, where $t = \inf_{1 \leqq n < \omega} \alpha_n > 0$. Then $\lambda(0) = t$, and no subset A of U is Lebesgue measurable. On the other hand, if $t = 0$, then $\lambda(0) = 0$ and all subsets of U are Lebesgue measurable.] *E. Hewitt* (Seattle, Wash.).

Császár, Ákos. Sur les nombres de Lipschitz approximatifs. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 211–214 (1950).

Let $F(x)$ be a finite real function of the real variable x. According to A. S. Besicovitch [Math. Z. **30**, 514–519 (1929)] the Lipschitz numbers of order α ($0 < \alpha < 1$) of $F(x)$ at x_0 are defined as

$$\overline{L}_\alpha^+ F(x_0) = \limsup_{h \to 0+} \{F(x_0+h) - F(x_0)\}/h^\alpha$$

and analogously $\overline{L}_\alpha^- F(x_0)$, $\underline{L}_\alpha^+ F(x_0)$, $\underline{L}_\alpha^- F(x_0)$. The author defines also the approximate Lipschitz numbers $\bar{\Lambda}_\alpha^+ F(x_0)$, $\bar{\Lambda}_\alpha^- F(x_0)$, $\underline{\Lambda}_\alpha^+ F(x_0)$, $\underline{\Lambda}_\alpha^- F(x_0)$ in the same manner, just replacing the upper (and lower) limits by the approximate upper (and lower) limits. Besicovitch [loc. cit.] proved that for every measurable function $F(x)$ the set of points x at which $\overline{L}_\alpha^- F(x) < \overline{L}_\alpha^+ F(x) < +\infty$ holds has measure zero. The author gives a simpler proof for this theorem, valid also for nonmeasurable functions $F(x)$. Moreover, if $F(x)$ is measurable, the author also proves the analogous theorem for the approximate Lipschitz numbers. *A. Rosenthal.*

Haupt, Otto. Zum Beweise des Lebesgueschen Ableitungssatzes. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 171–174 (1949).

This paper contains additional remarks on a previous article of the author and C. Pauc [Arch. Math. **1**, 23–28 (1948); these Rev. **10**, 359; see the definitions given in that review]. The author generalizes the conditions of the previous article, replacing axioms (U) and (E) by the following weaker combination of them. Axiom (UE): In $\mathfrak{R}$ there exists a σ-ring $\mathfrak{O}$ which (1) is a "Borel generator" of $\mathfrak{R}$ and (2) possesses the following property: Let $G \in \mathfrak{O}$ be given; then for almost every point $y \in G$ every $\mathfrak{B}_y$ is confinal with a subsystem of $\mathfrak{B}_y$ which is entirely contained in G. Moreover, the author gives sufficient conditions for the upper and lower derivates $\bar{D}F$ and $\underline{D}F$ to be m'-measurable if $m' | \mathfrak{R}'$ designates the complete extension of the originally given measure $m | \mathfrak{R}$. The set function F is not necessarily totally additive. Finally, the author analyzes differentiation with respect to nets. *A. Rosenthal* (Lafayette, Ind.).

Haupt, Otto, und Pauc, Christian Y. Über die Erweiterung eines Inhaltes zu einem Masse. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 247–253 (1949).

"Content" (Inhalt) $i | V$ here means a nonnegative additive set function i defined on a Boolean algebra V, "measure" $i | W$ means a nonnegative, totally additive set function i defined on a Boolean σ-algebra W. Here V and W have to

be subsystems of a fixed Boolean σ-algebra U. The authors discuss the extension of a totally additive $i|V$ to a complete measure $i|V''$. This question has been already previously investigated by several mathematicians. Here the authors perform this extension in two definite steps: (1) the extension of $i|V$ to a Borel measure $i|V'$, where V' is the smallest Boolean σ-algebra over V (in U); (2) the extension of the Borel measure $i|V'$ to the Lebesgue measure $i|V''$, i.e., the extension to a complete measure. In order to obtain the Borel extension (1), the authors first discuss the simple case where $i|V$ is "totally positive," i.e., where $i|V$ and the corresponding Lebesgue measure (provided that it exists) vanish only for the zero-element. If $i|V$ is totally positive and totally additive (in U), then $V_{\sigma\delta}=V_{b\sigma}=V'=V''$ holds. By means of this simple case the authors are then able to handle the general case also. *A. Rosenthal.*

Haupt, Otto, et Pauc, Christian. Mesure et topologie adaptées. Espaces mesurés topologiques. C. R. Acad. Sci. Paris **230**, 711–712 (1950).

A measure m is called normally adapted to a topological space X if (I) it is a regular Borel measure [see P. R. Halmos, Measure Theory, Van Nostrand, New York, 1950; these Rev. 11, 504]; (II) X is compact; (III) X has a basis of open sets whose frontiers are null sets ("null-frontier" sets). Samples of results announced are as follows. (1) If m is normally adapted to X then every finite covering by open sets has a refinement of disjoint null-frontier sets. (2) If m and n are normally adapted to X and Y, respectively, then the completion of the product measures is normally adapted to $X\times Y$. *R. Arens* (Los Angeles, Calif.).

Pauc, Christian. Intégrale de partition et intégrale topologique. Familles dérivantes topologiques. C. R. Acad. Sci. Paris **230**, 810–811 (1950).

This paper is in part a continuation of an earlier note [see the preceding review]. There is defined a Riemann-type integral over a topological measure space ("measure normally adapted to topology"; cf. reference above). The value of the integral is related to the Jordan content of the ordinate set and the classical condition on the measure of the discontinuities is extended. The second part announces extensions of the theory of differentiation of set functions. The differentiating family of sets is supposed to be a basis, and the measure assumed to be complete, the notion of "smallness" being referred to that of uniform structure. Certain theorems of equivalence of de Possel [same C. R. **201**, 579–581 (1935)] are available and permit the formulation of a result generalizing a proposition of Busemann and Feller [Fund. Math. **22**, 226–256 (1934)] to this: a necessary and sufficient condition that every measurable set has density 1 almost everywhere on itself is that for every α between 0 and 1 and any measurable M, the set $\bigcap S_{\alpha,\mathfrak{u}}$ differs from M by a null set, where each $S_{\alpha,\mathfrak{u}}$ consists of the union of those V of the differentiating family for which $m(VM)>\alpha m(V)$ and V is $\mathfrak{u}$-small, $\mathfrak{u}$ running through the uniform structure. *R. Arens* (Los Angeles, Calif.).

Deniston, R. F. Existence of Stieltjes integrals. I. Nederl. Akad. Wetensch., Proc. **52**, 1111–1119 = Indagationes Math. **11**, 385–393 (1949).

Deniston, R. F. Existence of Stieltjes integrals. II. Nederl. Akad. Wetensch., Proc. **52**, 1120–1128 = Indagationes Math. **11**, 394–402 (1949).

Ce mémoire apporte de nouvelles conditions d'existence pour les intégrales prises au sens de Stieltjes modifié. On considère des sommes de quatre types différents: $S=\sum f(\xi_i)[g(x_{i+1})-g(x_i)]$, où (A) $x_{i+1}\leqq\xi_i\leqq x_i$ (sommes ordinaires), (B) $\xi_i=x_i$, (C) $\xi_i=x_{i+1}$ (sommes de Cauchy à gauche et à droite), (D) $x_i<\xi_i<x_{i+1}$ (sommes de Dushnik). La limite des S est prise soit au sens ordinaire (O), soit au sens de Pollard (P), c'est-à-dire qu'à tout ϵ correspond une division D telle que pour toute subdivision de D, S diffère de sa limite de moins de ϵ. On a donc 8 types d'intégrales: AO, AP, BO, BP, CO, CP, DO, DP. On suppose que f soit bornée, et que g soit à variation bornée dans l'intervalle considéré. On introduit la "fonction de continuité" g_c de g égale à g diminuée de sa fonction des sauts. L'auteur indique que le cas AO a été étudie par Bliss [Proc. Nat. Acad. Sci. U. S. A. **3**, 633–637 (1917)], les cas BO et CO par Schärf [Portugaliae Math. **4**, 73–118 (1944); ces Rev. **6**, 120], et il donne pour chacun des autres cas les deux conditions nécessaires et suffisantes d'existence suivantes. Cas AP. (a) En tout point où g est discontinue à gauche (ou à droite), f est continue à gauche (ou à droite). (b) L'ensemble des points de discontinuité de f est de mesure nulle par rapport à g_c. Cas BP. (a) En tout point où g est discontinue à gauche, f a une limite à gauche. (b) L'ensemble des points ou f est discontinue à gauche est de mesure nulle par rapport à g_c. Cas CP. Analogue à BP. Cas DO. (a) En tout point, ou bien f est continue, ou bien g est continue, ou bien f et g peuvent être rendues continues en modifiant leur valeur en ce point. (b) Comme AP. Cas DP. (a) En tout point où g est discontinue à gauche (ou à droite), f a une limite à gauche (ou à droite). (b) Comme AP. Les démonstrations font appel à la notion de pseudo-additivité d'une fonction d'intervalle introduite par Getchell [Bull. Amer. Math. Soc. **41**, 413–418 (1935)]. *R. de Possel* (Alger).

Ridder, J. Stieltjessche Integrale. Nederl. Akad. Wetensch., Proc. **52**, 1129–1134 = Indagationes Math. **11**, 403–408 (1949).

L'auteur reprend les définitions des 8 intégrales de Deniston [voir l'analyse ci-dessus], mais il remplace les sommes S par des e-sommes définies plus bas. Voici ses résultats, énonces en supposant pour simplifier $g(x)$ monotone. Pour que l'une des 8 intégrales définies avec les e-sommes existe, il faut et il suffit que les points de discontinuité de $f(x)$ où $g(x)$ est continue forment un ensemble de mesure nulle par rapport à g. La valeur obtenue coincide avec celle de l'intégrale de Lebesgue-Stieltjes qui existe toujours dans ces conditions. Voici la définition d'une e-somme. Soit $T=\sum f(\bar{\xi}_i)[g(\overline{x_{i+1}})-g(\bar{x}_i)]$ où $\bar{\xi}_i$, $\bar{x}_i$ représente soit un point, soit un symbole de la forme $\overline{x\pm 0}$, avec les conventions suivantes: $g(\overline{x_0-0})=\lim_{x\uparrow x_0} g(x)$, $g(\overline{x_0+0})=\lim_{x\downarrow x_0} g(x)$, $f(\overline{x_0-0})=f(x_0)$, $f(\overline{x_0+0})=f(x_0)$. La suite Σ des $\bar{x}_i$ vérifie les conditions $a=\bar{x}_1<\bar{x}_2<\cdots<\bar{x}_n=b$, avec les conventions $\overline{x-0}<x<\overline{x+0}$ en tout point x où g est discontinue. La somme T est dite une e-somme si les conditions suivantes sont vérifiées: (1) pour tout point x tel que $g(\overline{x+0})-g(\overline{x-0})\geqq e$, les 3 points $\overline{x-0}$, x, $\overline{x+0}$ appartiennent à Σ. (2) Pour toute paire $(\bar{x}_i, \overline{x_{i+1}})$ qui n'est pas de la forme $(\overline{x-0}, x)$ ou $(x, \overline{x+0})$, on a $g(\overline{x_{i+1}})-g(\bar{x}_i)<e$. *R. de Possel* (Alger).

Reichelderfer, Paul V. On the semi-continuity of double integrals. Trans. Amer. Math. Soc. **67**, 275–285 (1949).

Let us consider the class C of all continuous oriented surfaces S of the space $x=(x_1, x_2, x_3)$ having an eAC representation $x=x(u)$, $u\in R$, R a Jordan region. If $x=x(u)$, $u\in R$, is such a representation of a surface S, then the generalized Jacobians $J_r(u)$, $r=1, 2, 3$, exist almost everywhere in R_0

and the Lebesgue area is given by the classical integral $L(S)=\int\int_{R_0}|J|du$, $J=(J_1, J_2, J_3)$ [T. Radó and the author, same Trans. 49, 258–307 (1941); these Rev. 2, 257; the reviewer, Atti Accad. Italia. Mem. Cl. Sci. Fís. Mat. Nat. 12, 1305–1397 (1942); 13, 1323–1481 (1943); 14, 891–951 (1944); Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 350–365 (1942); these Rev. 8, 142]. Let $f(x, X)$, $X=(X_1, X_2, X_3)$, be a function of x, X continuous with its first derivatives f_r with respect to X_r, $r=1, 2, 3$, and $f(x, tX)=tf(x, X)$ for any $t\geqq 0$. Let E be the Weierstrass function $E=E(x, X, \bar{X})=f(x, \bar{X})-\sum \bar{X}_r f_r(x, X)$ and $I(S)$ the integral $I(S)=\int\int_S f(x, X)du$ on the surface S. The main result is that if (a) $S\varepsilon C$, (b) $f(x, X)\geqq 0$ for any vector X and any x of an open set $A\supset S$, (c) $E[x(u), J(u), X]\geqq 0$ for almost all $u\varepsilon R_0$ such that $|J|\neq 0$ and any vector X, then $I(S)$ is a lower semicontinuous functional on S with respect to all surfaces $S'\varepsilon C$. As the author recalls, the class C coincides with the class of all surfaces S with $L(S)<+\infty$ by a result of the reviewer [Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 101, 377–403 (1942); Ann. Mat. Pura Appl. (4) 26, 301–374 (1947); these Rev. 8, 258; 10, 362]. The present result extends a previous paper of T. Radó [same Trans. 51, 336–361 (1942); these Rev. 3, 229] from which notations and methods are utilized. [An analogous investigation has been carried out by the reviewer utilizing quite different methods, Ann. Scuola Norm. Super. Pisa (2) 13 (1944), 77–117 (1948); 14 (1945), 47–79 (1948); these Rev. 9, 505; 10, 259.] *L. Cesari* (Lafayette, Ind.).

Rado, T., and Reichelderfer, P. V. On n-dimensional concepts of bounded variation, absolute continuity and generalized Jacobian. Proc. Nat. Acad. Sci. U. S. A. 35, 678–681 (1949).

Let f be a one-valued continuous mapping (not necessarily one-to-one) of the Euclidean space U^n with points $u=(u_1, \cdots, u_n)$ into the Euclidean space X^n with points $x=(x_1, \cdots, x_n)$. An n-dimensional concept of essential absolute multiplicity function $k(x, f, U^n)$ has recently been proposed by H. Federer in terms of relative Čech cohomology groups as an extension of the same notion introduced in a more elementary way in the case $n=2$ by T. Radó and by the reviewer. A mapping f is called eBV if $k(x, f, U^n)$ is L-integrable.

The authors introduce notions of positive, negative and signed multiplicity functions k^+, k^-, μ, where $k=k^+ + k^-$, $\mu=k^+ - k^-$, $k^+\geqq 0$, $k^-\geqq 0$. For any interval $q\subset U^n$ let $F(q)$ be the L-integral of $\mu(x, f, q)$ in the space X^n. Thus the generalized Jacobian $J(u)$, $u\varepsilon U^n$, is, almost everywhere in U^n, the derivative of $F(q)$ as an interval function. By suitable extension of the notion of the set E, $E\subset U^n$, covered by all the essential maximal model continua of f, the mapping f is called eAC if and only if for each measurable subset $I\subset U^n$ the relation $|EI|=0$ implies $|f(EI)|=0$. Applications to the problem of the transformation of nth integrals are to be given in a future more detailed paper [for $n=2$ see T. Radó and P. Reichelderfer, Trans. Amer. Math. Soc. 49, 258–307 (1941); the reviewer, Ann. Mat. Pura Appl. (4) 27, 321–374 (1948); these Rev. 2, 257; 11, 20].

L. Cesari (Lafayette, Ind.).

Youngs, J. W. T. Topological methods in the theory of Lebesgue area. Bull. Amer. Math. Soc. 56, 17–31 (1950).

This address is a general outline of the recent developments of Lebesgue area theory. The following new results of the author are also announced. (I) Let $f_1: X\to E^3$, $f_2: X\to E^3$, be given continuous transformations from any continuous compact 2-manifold X into the Euclidean 3-space E^3. Then f_1 and f_2 are representations of the same Fréchet surface if and only if: (a) there is a monotone-light factorization lm_i of m_i with common middle space $\mathfrak{X}$, $i=1, 2$; (b) if $\dot{m}_i=m_i|\dot{X}$ ($\dot{X}$ boundary of X), then there is a monotone-light factorization $\lambda\mu_i$ of $\dot{m}_i$ with common middle space Ξ, $i=1, 2$; (c) if $m_i^*: H^2(\mathfrak{X}, \dot{\mathfrak{X}})\to H^2(X, \dot{X})$ and $\mu_i^*: H^1(\Xi)\to H^1(\dot{X})$ are the homomorphisms induced by the mappings m_i and μ_i, respectively, $i=1, 2$, and $\delta: H^1(\dot{X})\to H^2(X, \dot{X})$, then there is an automorphism $\eta: H^2(X, \dot{X})\approx H^2(X, \dot{X})$ such that $\delta\mu_1^*=\eta\delta\mu_2^*$, $m_1^*=\eta m_2^*$. This result, involving, as far as it concerns condition (c), the Čech cohomology theory, solves the topological representation problem, that is, the topological characterization of the Fréchet equivalence. (II) If $f: X\to E^3$ is a given transformation (see above) and lm is a monotone-light factorization of f with middle space $\mathfrak{X}$, then f has Lebesgue area zero if and only if $\dim \mathfrak{X}<2$. This result characterizes the surfaces of zero Lebesgue area, generalizing a result of T. Radó [Length and Area, Amer. Math. Soc. Colloquium Publ., v. 30, New York, 1948; these Rev. 9, 505] for the case where X is a 2-cell or a 2-sphere.

L. Cesari (Lafayette, Ind.).

Cesari, Lamberto. On the representation of surfaces. Amer. J. Math. 72, 335–346 (1950).

In the theory of the Lebesgue area of continuous surfaces $S: x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $(u, v)\varepsilon Q$: $0\leqq u, v\leqq 1$, there arise the following two questions [for terminology, see the reviewer's book: Length and Area, Amer. Math. Soc. Colloquium Publ., v. 30, New York, 1948; these Rev. 9, 505]. (1) If the Lebesgue area $A(S)$ of S is finite, does there exist a representation of S where $A(S)$ is equal to the value of the classical area-integral calculated in terms of essential generalized Jacobians? (2) If $A(S)<\infty$, does there exist a representation of S where $A(S)$ is equal to the classical area-integral in terms of ordinary Jacobians? In recent papers, the author gave an affirmative answer to question (1). The present paper furnishes an affirmative answer to question (2).

T. Radó (Columbus, Ohio).

Helsel, R. G. A note on convergence in area. Proc. Amer. Math. Soc. 1, 23–24 (1950).

Let $T_n: x=x_n(u, v)$, $(u, v)\varepsilon Q$, Q the unit square, be a sequence of continuous mappings approaching in the sense of Fréchet a given continuous mapping T_0: $x=x_0(u, v)$, $(u, v)\varepsilon Q$, where $x=(x^1, x^2, x^3)$ is any point of the Euclidean space E_3. Let O be the point $x=0$, U the sphere $|x|=1$, p any point of U, $E_2(p)$ any plane of E_3 perpendicular to the segment O_p, T_p the projection of E_3 into $E_2(p)$. Then T_pT_n, T_pT_0 are continuous plane mappings of Q into $E_2(p)$. Let $L(T)$ be the Lebesgue area of any mapping T. By application of recent results of the author, of T. Radó and of the reviewer the following is proved: $L(T_n)\to L(T_0)$ if and only if $L(T_pT_n)\to L(T_pT_0)$ for each $p\varepsilon U$. This result extends to Lebesgue area of Fréchet surfaces a previous result for length of curves [M. C. Ayer and T. Radó, Bull. Amer. Math. Soc. 54, 533–539 (1948); these Rev. 10, 24].

L. Cesari (Lafayette, Ind.).

Theory of Functions of Complex Variables

***Hornich, Hans. Lehrbuch der Funktionentheorie.** Springer-Verlag, Wien, 1950. viii+216 pp. $4.70; bound, $5.20.

This is a sound introductory textbook covering classical material. More attention than usual is devoted to conformal mapping; the beta function is introduced via the mapping of a triangle on a half-plane; the Riemann mapping theorem and the "small" Picard theorem are proved. There is a long chapter on algebraic functions and their integrals, leading up to doubly periodic functions. Topological difficulties are passed over. *R. P. Boas, Jr.* (Evanston, Ill.).

Hervé, Michel. Sur l'itération dans un domaine multiplement connexe. C. R. Acad. Sci. Paris **230**, 707–708 (1950).

Soit D un domaine du plan de la variable complexe z d'ordre de connexion p, dont la frontière se compose de p continus non ponctuels disjoints; le disque $|Z|<1$ est représenté sur la surface universelle de recouvrement de D par une fonction Φ automorphe relativement à un groupe $\mathfrak{G}$ de substitutions S. L'auteur considère la famille $\mathfrak{F}$ des fonctions $f(z)$ analytiques uniformes dans D, dont les valeurs appartiennent à D et qui ne représentent pas D biunivoquement sur lui-même; à $f(z)$ il associe la fonction $F(Z)=\Phi^{-1}[f(\Phi)]$. Si $F_n(Z)$ est l'itérée d'ordre n de $F(Z)$, on a, pour toute S de $\mathfrak{G}$ et pour $n>n_0$ (n_0 indépendant de f), $F_n(SZ)\equiv F_n(Z)$. [Comparer M. Heins, Amer. J. Math. **63**, 461–480 (1941); Duke Math. J. **8**, 312–316 (1941); ces Rev. **2**, 275; **3**, 81.] Pour $p=2$, le groupe $\mathfrak{G}$ se réduit aux itérées d'une seule substitution S et (1) dévient $F(SZ)\equiv F(Z)$; la famille $\mathfrak{F}$ ne peut pas être partagée en sous-classes, tandis qu'un exemple montre qu'il n'en est plus de même pour $p=3$. Pour $p=2$, l'auteur donne explicitement la constante du point fixe z_0, introduite par Aumann et Carathéodory [Math. Ann. **109**, 756–763 (1934)], égale à $\sup|f'(z_0)|$, et il détermine les fonctions extrémales pour lesquelles cette borne est atteinte. Il étudie les valeurs prises par ces fonctions extrémales, notamment dans le cas où les deux continus frontières de D sont des courbes de Jordan. *G. Valiron* (Paris).

Rådström, Hans. Zeros of successive derivatives. Ark. Mat. **1**, 101–139 (1950).

Let $f(z)$ be an analytic function, and let D be any bounded simply connected open set on which $f(z)$ is regular, and which contains no limit-point of the zeros of the derivatives of $f(z)$; D_- denotes a closed subset of D. Let $a_n(x)=f^{(n)}(x)/n!$; $g_{nm}(x)=g_n(x)=\{a_n(x)\}^{1/n}$, where m refers to the branch of the nth root; $g=g(x)=\liminf|g_n(x)|$; $h_n(x)=a_{n+1}(x)/a_n(x)$; $R=R(x)$ is the distance from x to the nearest singular point of $f(z)$. Several results on the convergence and the limits of sequences $\{g_{n_k m_k}(z)\}$ and $\{h_{n_k}(z)\}$ in D are proved. (1) The family $\{g_{nm}(z)\}$ is normal in D (that is, every sequence of functions $\{g_{n_k m_k}(z)\}$ contains a subsequence, uniformly convergent in every D_-). If $g>0$ at a point $x\in D$, then $\{h_n(z)\}$ is a normal family in D_- ($n>n_0$). (2) If $g_{n_k m_k}(z)\to\varphi(z)$ in D, $\varphi(z)\not\equiv 0$, then $h_{n_k}(z)\to\varphi'(z)/\varphi(z)$ in D. If $\{|g_{n_k}(z)|\}$ converges in D to a limit, not identically 0, then all limit-functions of the family $\{g_{n_k m_k}(z)\}$ are of the form $\eta\cdot\varphi(z)$, $|\eta|=1$, and $h_{n_k}(z)\to\varphi'(z)/\varphi(z)$. (3) Is it true that the convergence of $\{|g_{n_k}(x)|\}$ at a point $x\in D$ implies that $\{h_{n_k}(z)\}$ converges in D? At x? Or does $\{|h_{n_k}(x)|\}$ converge at least? Several partial answers are given. If $\{|g_{n_k}(x)|\}$ is maximal, that is, $|g_{n_k}(x)|\to 1/R$, and if $x\in D$, then $\limsup|h_{n_k}(x)|\leqq 1/R$ [see also (7)]. Some weaker inequalities for $|h_{n_k}(x)|$ are obtained in the case $|g_{n_k}(x)|\to L_1<1/R$, $x\in D$. (4) If $x\in D$, $g(x)>0$, then the limit-points of $\{|g_n(x)|\}$ form an interval. Similarly, if $x\in D$, $\liminf|h_n(x)|>0$, then the limit-points of $\{h_n(x)\}$ form a connected set. This follows from the stronger result that in this case the set of points $n\{h_{n+1}(z)-h_n(z)\}$ is bounded ($z\in D_-$, $n>n_0$).

Other results are given as gap theorems. (5) If $g<1/R$ at $x\in D$, then the power-series for $f(z)$ at x has Ostrowski gaps (that is, there are numbers $c>1$ and $\gamma<1/R$ and a sequence $n_k\to\infty$ such that $|a_n(x)|<\gamma^n$ for $n_k\leqq n<cn_k$). (6) If $g=0$ at $x\in D$, then $f(z)$ is the sum of an entire function and a function $F(z)$ whose power series at x has large gaps (that is, there exist sequences $n_k\to\infty$, $c_k\to\infty$, such that $F^{(n)}(x)=0$, $n_k\leqq n<c_k n_k$).

Mention is made of the singularities of $f(z)$ in the following results. (7) [Compare (3).] If $\{|g_{n_k}(x)|\}$ is maximal, $x\in D$, then all limit-points of $\{h_{n_k}(x)\}$ belong to the closed convex set spanned by the points $1/(s-x)$, where s runs through the singular points of $f(z)$ on $|z-x|=R$. This implies that the singular points of $f(z)$ on $|z-x|=R$, $x\in D$, are the points $x+1/L$, where L runs through the limit-points of those sequences $\{h_{n_k}(x)\}$ for which both $|g_{n_k}(x)|$ and $|h_{n_k}(x)|\to 1/R$. (8) If $|g_n(z)|$ converges for all $z\in D$ to a nonzero limit then the circle of convergence at $x\in D$ contains only one singular point. Conversely, if the circle of convergence at the regular point x contains only one singular point, which is a pole (hence $\{|g_n(x)|\}$ converges to $g(x)>0$) then $x\in D$. This result and (2) are used to prove Pólya's result on the limit-points of the zeros of the derivatives of meromorphic functions [Math. Z. **12**, 36–60 (1922)].

The above results supply sufficient conditions for a regular point to be a limit-point of the zeros of the derivatives of $f(z)$. Most of those conditions will not be necessary, as follows from: (9) if $x\in D$, $R<\infty$, then there exist numbers ω_n, $|\omega_n|=1$, such that x is a limit-point of the zeros of the derivatives of $\sum\omega_n a_n(x)(z-x)^n$. (10) The following is a necessary and sufficient condition: a regular point x with $g>0$ is a limit-point of the zeros of the derivatives of $f(z)$ if and only if there is a $B>0$ such that $|h_n^{(m)}(x)|<m!B^m$ for $n>n_0$ and all m. *J. Korevaar* (Lafayette, Ind.).

Alenicyn, Yu. E. On univalent majorants. Mat. Sbornik N.S. **26(68)**, 57–74 (1950). (Russian)

A function $G(z)$, regular and univalent in the circle $|z|<R$, is said to be a univalent majorant of a function $g(z)$ regular in $|z|<R$ if $G(0)=g(0)$ and if the set of values assumed by $g(z)$ for $|z|<R$ is included in the set of values assumed there by $G(z)$; in symbols: $g(z)\dashv G(z)$ for $|z|<R$. The problem investigated by R. M. Robinson [Trans. Amer. Math. Soc. **61**, 1–35 (1947); these Rev. **8**, 370] is to determine the largest circle $|z|\leqq r$ such that the condition $g(z)\dashv G(z)$ for $|z|<1$ implies $\mathfrak{F}g(z)\dashv\mathfrak{F}G(z)$ for $|z|\leqq r$, where $G(z)$ is a preassigned majorant and $\mathfrak{F}$ a linear operator. In the present paper the analogous problem is studied for functions whose expansions are of special type and for operators $\mathfrak{F}$ which are not necessarily linear. Let $g(z)$ be regular in $|z|<1$. Denote by F_s an operator which satisfies the two conditions: (1) when applied on $g(z)$ it gives a function of z regular in $|z|<1$; and (2) if $g(z)=G(\zeta(z))$ in $|z|<1$, where $\zeta(z)$ is regular in $|z|<1$ and $|\zeta(z)|<1$ there, then $F_\zeta G(\zeta)\equiv F_z g(z)$, $\zeta=\zeta(z)$, $|z|<1$. The operator $\mathfrak{F}$ is now defined by $\mathfrak{F}_z g(z)=F_z g(z)+zg'(z)$. A basic result is that if $g_n(z)=a_0+a_1z+a_{n+1}z^{n+1}+\cdots$ is regular and $G_n(z)=A_0+A_1z+A_{n+1}z^{n+1}+\cdots$ is regular and univalent in $|z|<1$ (the dots do not imply the necessary existence of

gaps), then $g_n(z) \prec G_n(z)$ for $|z|<1$ implies the relation $\mathfrak{F}g_n(z) \prec \mathfrak{F}G_n(z)$ for $|z| \leqq r$ if, and only if, the two conditions are satisfied: (1) $\mathfrak{F}G_n(z)$ is univalent in $|z| \leqq r$; (2) for all ζ and Z with $|\zeta| \leqq |Z| = r$ the inequality

$$\left|\frac{\mathfrak{F}G_n(Z)-\mathfrak{F}G_n(\zeta)}{G_n'(\zeta)}\right| \geqq \frac{nr^{n-1}(r^2-|\zeta|^2)}{1-r^{2n}}$$

must hold. For $n=1$ and $G(z)=z+\cdots$ the following result is obtained. If $g(z) \prec G(z)$ for $|z|<1$, then

$$\mu g(z)+zg'(z) \prec \mu G(z)+zG'(z)$$

in the circle $|z| \leqq r_\mu$, where $r_\mu=(1-\mu)^{-1}\{(8+\mu^2)^{\frac{1}{2}}-3\}$ for $-2.03\cdots \leqq \mu < -1$ and the negative of this for $-1<\mu \leqq 1.39\cdots$, but not always for a larger circle. For $\mu=0$ and $\mu=1$ this reduces to Robinson's results. A number of other results of this character are obtained.

W. Seidel (Rochester, N. Y.).

Ter-Mikaèlyan, T. M. A lower bound for the harmonic measure of a set on certain rectifiable curves. Doklady Akad. Nauk SSSR (N.S.) **71**, 625–628 (1950). (Russian)

Let $D(q)$ be an arbitrary finite simply connected region containing the circle $|z|<1$ in its interior, bounded by a rectifiable curve Γ, and satisfying the condition that the ratio of the length $L(\gamma)$ of an arbitrary arc γ of Γ to the length of the corresponding chord $d(\gamma)$ is bounded by a constant q which is independent of γ. It is proved that under a conformal map $w=F(z)$, $F(0)=0$, of the region $D(q)$ on the circle $|w|<1$ there corresponds to an arbitrary measurable set E_z on Γ of measure ϵ a set E_w on $|w|=1$ whose measure $mE_w > M\epsilon^\delta$, where δ is a constant greater than one which depends on q alone and M is a constant depending on q and the length of Γ. A similar inequality in the opposite direction has been obtained by M. A. Lavrentieff [Rec. Math. [Mat. Sbornik] N.S. **1(43)**, 815–846 (1936)].

W. Seidel (Rochester, N. Y.).

Nehari, Zeev. On bounded analytic functions. Proc. Amer. Math. Soc. **1**, 268–275 (1950).

The author gives a new derivation of a generalization of Schwarz's lemma [Garabedian, Trans. Amer. Math. Soc. **67**, 1–35 (1949); Ahlfors, Duke Math. J. **14**, 1–11 (1947); these Rev. **11**, 340; **9**, 24]. Let D be a multiply-connected domain bounded by Γ, let $\zeta \varepsilon D$, and let

$$l_\lambda(z) = -\frac{e^{i\Delta}}{(z-\zeta)^2} - \frac{\alpha}{z-\zeta} + \text{regular terms},$$

with $il_\lambda(z)dz \leqq 0$, $z \varepsilon \Gamma$, be the function with such a pole which minimizes α. Then

$$K(z,\zeta)=\tfrac{1}{4}\pi^{-1}ie^{i\Delta}(\sqrt{l_\lambda}+i\sqrt{l_{\lambda+\pi/2}}),$$
$$L(z,\zeta)=-\tfrac{1}{4}\pi^{-1}ie^{-i\Delta}(\sqrt{l_\lambda}-i\sqrt{l_{\lambda+\pi/2}})$$

are Szegö kernel functions and are independent of λ. The regular function $F(z)=K(z,\zeta)/L(z,\zeta)$ has the largest derivative at $z=\zeta$ of all functions bounded in modulus by 1 in D.

P. R. Garabedian (Stanford University, Calif.).

Nehari, Zeev. Conformal mapping of open Riemann surfaces. Trans. Amer. Math. Soc. **68**, 258–277 (1950).

The main purpose of the paper is to prove the existence of certain canonical mappings of Riemann surfaces R of finite genus g and a finite number of contours. The proofs make essential use of the Abelian integrals on the Schottky "double" of the surface, and accordingly the theory of Schottky integrals and their connection with the Bergman kernel functions is discussed in considerable detail. The first existence theorem asserts that R can be mapped onto a plane with $g+1$ sheets bounded by slits parallel to the real axis and prescribed poles $\zeta_0, \zeta_1, \cdots, \zeta_g$ with a given residue at ζ_0. The mapping may degenerate, in which case the plane has fewer sheets. The second theorem treats the corresponding question for concentric or radial slits. In this case it is only shown that one pole and one zero can be prescribed. Uniqueness is not expected and the problem of a more complete determination of the mapping is left open. The proof of the first theorem is elementary while the second is made to depend on the solution of Jacobi's inversion problem.

L. Ahlfors (Cambridge, Mass.).

Komatu, Yûsaku. Existence theorem of conformal mapping of doubly-connected domains. Kôdai Math. Sem. Rep., no. **5–6**, 3–4 (1949).

A proof is given for the classical theorem concerning the conformal mapping of a doubly-connected region onto an annulus. Use is made of the Riemann mapping theorem for simply-connected regions to reduce the problem to the case where the region considered is bounded by a circle and an analytic Jordan curve. The proof is completed by extremal considerations for doubly-connected regions of the indicated reduced type.

M. Heins (Providence, R. I.).

Fourès, Léonce. Sur les surfaces de Riemann à arbre topologique régulièrement ramifié. C. R. Acad. Sci. Paris **230**, 353–355 (1950).

The topological tree of a Riemann surface with branch-points over a finite number of points is said to be regularly ramified if there are only a finite number of inequivalent nodes under the automorphisms of the tree. The author gives some careful definitions and discusses the conditions under which the surface can be considered as a smooth covering surface of a closed surface.

L. Ahlfors.

Garabedian, P. R. The sharp form of the principle of hyperbolic measure. Ann. of Math. (2) **51**, 360–379 (1950).

Let D and Δ be regions in the z- and w-planes of connectivity n and m. Consider all analytic and single-valued functions $w=F(z)$ in D with values in Δ which satisfy a condition $F(z_0)=w_0$. The problem is to characterize the extremal functions $F(z)$ which make $|F'(z_0)|$ a maximum. The author's result is this: every extremal function maps D upon a relatively unbounded covering surface S of Δ with at most $2n-2$ branch-points; S may have infinitely many sheets. The proof is by variation of the Riemann surface S. The variation must be chosen so that the varied surface S^*, with fixed w_0, is conformally equivalent with S. This means that $3n-4$ moduli must have prescribed values. As moduli the author chooses the $n-1$ conjugate periods of the Green's function G with pole at w_0, the values of G at its $n-1$ critical points, and the increments of the conjugate function along $n-2$ paths joining the critical points. For small variations it can hardly be doubted that these quantities determine the conformal type, but no formal proof is given. The variation of the boundary is carried out by Schiffer's method [M. Schiffer, Amer. J. Math. **68**, 417–448 (1946); these Rev. **8**, 325] and leads to the asserted behavior on the boundary. The variation of the branch-points shows that they are zeros of a quadratic differential, and this property limits their number. In the case $m=2$ a certain limited uniqueness is proved. The results can also be generalized to Riemann surfaces of positive finite genus, but the details are not worked out. No attempt is made to determine when S must have a finite number of sheets.

L. Ahlfors.

Temlyakov, A. A. On harmonic functions and functions of two complex variables with analytic determining functions. Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 240–245 (1950). (Russian)

This paper is a very brief summary of the author's thesis. The author considers a harmonic function

$$F(x, y, z) = (2\pi)^{-1} \int_0^{2\pi} f(u, e^{it}) dt,$$

where $f(u, e^{it})$ is an integral function of

$$u = x + i(y \cos t + z \sin t),$$

and where F in polar coordinates possesses the development

$$F)R, \theta, \varphi) = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} (A_{nm}^{(1)} e^{im\varphi} + A_{nm}^{(2)} e^{-im\varphi}) P_n^{(m)}(\cos \varphi) \right] R^n.$$

The number $\rho = \limsup_{R \to \infty} \log \log M(R) / \log R$, where $M(R) = \sup |F(x, y, z)|$ for $x^2 + y^2 + z^2 = R^2$, is called the order of F. It is the greatest lower bound of the set of positive numbers k satisfying

$$|A_{nm}^{(1)}| < 2\{(2n+1)(n-m)!/(n+m)!\}^{\frac{1}{2}} (ek/n)^{n/k}.$$

If the function $f(u, e^{it})$ for a fixed value of t assumes the given value $\alpha + i\beta$ at the points $u_1, u_2, \cdots$, the series $\sum |u_\nu|^{-(\rho+\epsilon)}$ will converge when ϵ is positive. If, on the other hand, $\sum |u_\nu|^{\rho(t)}$ converges for two different choices of $\alpha + i\beta$, then $\rho \leqq \sup_{0 \leq t \leq 2\pi} \rho(t)$. The type σ of F is defined by $\sigma = \lim_{R \to \infty} R^{-\rho} \log M(R)$ and satisfies

$$\lim_{n \to \infty} \max_{0 \leq m \leq n} n^{1/\rho} \left\{ \frac{(n+m)!}{(n-m)!} |A_{nm}^{(1)}| \right\}^{1/(2n)} = (e\sigma\rho)^{1/\rho}.$$

The author states some results concerning continuation of F in the case where $f(u, e^{it})$ is meromorphic. Similar investigations are carried out for functions of two complex variables. For a function $F(x, y)$ analytic in a "hypercone" $|x| + |y| \leqq R$ we have a representation

$$F(x, y) = (2\pi)^{-1} \int_0^{2\pi} f(u, e^{it}) dt,$$

where $u = x + ye^{it}$ and

$$f(u, e^{it}) = \frac{1}{2\pi i} \int_0^1 dT \int_{|\xi| = R} \frac{\xi F(\xi T, \xi(1-T)e^{-it})}{(\xi - u)^2} d\xi.$$

If $F(x, y)$ is integral, its order ρ is given by

$$\rho = \limsup_{R \to \infty} \log \log M(R) / \log R,$$

where $M(R) = \sup |F(x, y)|$ for $|x| + |y| = R$ and ρ is the greatest lower bound of the set of numbers k satisfying $|a_{nm}| < (n+m)^{n+m} n^{-n} m^{-m} (ek/(n+m))^{(n+m)/k}$, where a_{nm} is the general coefficient of the power series $F(x, y) = \sum a_{nm} x^n y^m$. A connection with the zeros is found in this case too. If $F(x, y)$ is an arbitrary analytic function, the radius R of the "hypercone of convergence" is determined by the relation

$$R^{-1} = \lim_{n \to \infty} \left\{ \sum_{k=0}^{n} \frac{k!(n-k)!}{n!} |a_{k, n-k}| \right\}^{1/n}.$$

If the hypercone of convergence is the natural domain of definition of F, every point of the manifold $|x| = |y| = R$ is a singular point of the function $\sum \{n!m!/(n+m)!\} a_{nm} x^n y^m$. Some further theorems of a similar type are mentioned. Finally the author states some results concerning biharmonic functions, e.g., the following representation for a function $\Re F(x, y)$ in the hypercone $|x| + |y| \leqq R$: $\Re F(x, y) =$

$$\frac{1}{4\pi^2} \int_0^{2\pi} dt \int_0^1 dT \int_0^{2\pi} \Re \left[\frac{2z^2}{(z-u)^2} - 1 \right] \Re F(zT, ze^{-it}(1-T)) d\theta,$$

where $z = Re^{i\theta}$ and $u = x + ye^{it}$. No proofs are given.

H. Tornehave (Copenhagen).

Roncati, Franca. Un confronto tra il metodo del Sobrero e quello del Burgatti per lo studio degli integrali dell'equazione differenziale alle derivate parziali $\Delta^{(n)} u = 0$. Atti Accad. Ligure **5**, 331–337 (1949).

P. Burgatti [Boll. Un. Mat. Ital. (1) **1**, 8–12 (1922)] introduced the notion of conjugate real valued polyharmonic functions as a natural generalization of that of conjugate real valued harmonic functions, and considered complex valued functions (called analytic functions of order n) $u(x, y) + iv(x, y)$, where u and v are conjugate n-harmonic functions. For analytic functions of order 2, Burgatti [Boll. Un. Mat. Ital. (1) **2**, 87–93 (1923)] generalized the Cauchy integral formula to

$$(*) \quad f(z) = (2\pi i)^{-1} \int_C (\zeta - z)^{-1} f(\zeta) d\zeta - (4\pi i)^{-1} \int_C (\bar{\zeta} - \bar{z})(\zeta - z)^{-1} D_\zeta f(\zeta) d\zeta,$$

where $D_\zeta = \partial/\partial x + i\partial/\partial y$. L. Sobrero [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **6**, 1–64 (1934)], in dealing with the biharmonic equation $\Delta\Delta u = 0$, introduced a hypercomplex unit j such that $(1 + j^2)^2 = 0$, and considered analytic functions

$$f(z) = a(x, y) + jb(x, y) + j^2 c(x, y) + j^3 d(x, y),$$

of $z = x + jy$, x and y real. For these analytic functions Sobrero proved the Cauchy formula

$$(**) \quad f(z) = (2\pi i)^{-1} \int_C (\zeta - z)^{-1} f(\zeta) d\zeta,$$

where $i = \frac{1}{2}(3j + j^3)$, $z = x + jy$, and $\zeta = \xi + j\eta$. The author shows the equivalence of (*) and (**). *J. B. Diaz.*

Fourier Series and Generalizations, Integral Transforms

***Natanson, I. P. Konstruktivnaya teoriya funkcil. [Constructive Theory of Functions].** Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad, 1949. 688 pp.

Uniform approximation: Weierstrass's theorem; Algebraic polynomials of best approximation; Trigonometric polynomials of best approximation; Influence of the structural properties of a function on the degree of its approximation by trigonometric polynomials. Characteristic structural properties of a function and the general behavior of its best approximation by trigonometric polynomials. Connection between the structural properties of a function and its approximation by algebraic polynomials. Fourier series as a means of approximation. Fejér and Vallée Poussin sums. Best approximation of analytic functions. Properties of some analytic approximation methods. Quadratic approximation: Space $L_p^2(x)$. Orthogonal systems. Linear independence of systems of functions. General properties of orthogonal polynomials. Legendre polynomials. Jacobi polynomials. Moment problem for a finite interval. Case of an infinite interval. Integration and mechanical quadratures. Various aspects of interpolation. Results of a negative character. Convergence of interpolation processes. Some convergent processes related to interpolation. Mechanical quadratures. Supplementary material on the theory of mechanical quadratures. *Table of contents.*

Duffin, R. J. Representation of Fourier integrals as sums. II. Proc. Amer. Math. Soc. 1, 250–255 (1950).

The author returns to the identities of part I [Bull. Amer. Math. Soc. 51, 447–455 (1945); these Rev. 6, 266]. The formula quoted in the review of part I is established, with $(C, 1)$ summability replacing convergence in most places, for the cases (i) where $\phi(x)$ and $x^{-1}\phi(x)$ have derivatives of bounded variation vanishing at ∞, (ii) $\phi(x)$ is analytic and of slow enough growth in a sector around the positive real axis. The results depend on the following theorem on summable sine transforms, which appears to be new. Let $f'(x)$ have bounded variation in $(0, \infty)$ and $f'(\infty)=0$. Then $\int_0^\infty f(t)\sin\frac{1}{2}xt\,dt$ is summable $(C, 1)$ to $g(x)$, and $f(x)=\int_0^\infty g(t)\sin\frac{1}{2}xt\,dt$ for $x>0$. *R. P. Boas, Jr.*

Mohanty, R. On the summability $|R, \log w, 1|$ of a Fourier series. J. London Math. Soc. 25, 67–72 (1950).

It is shown, by an example, that summability $|R, \log w, 1|$ of a Fourier series $\frac{1}{2}a_0+\sum_1^\infty(a_k\cos kx+b_k\sin kx)=\sum_0^\infty A_k(x)$ is not locally conditioned. [Compare, for the corresponding result on $|C, 1|$, Bosanquet and Kestelman, Proc. London Math. Soc. (2) 45, 88–97 (1939).] As regards the series $\sum A_n(x)/\log(n+1)$, summability $|C, 1|$ is not locally conditioned, but summability $|R, \log w, 1|$ is.

W. W. Rogosinski (Newcastle-upon-Tyne).

Mohanty, R. The absolute Cesàro summability of some series associated with a Fourier series and its allied series. J. London Math. Soc. 25, 63–67 (1950).

Suppose that $f(x)$ is L, of period 2π, and has Fourier coefficients a_k, b_k. Let

$$A_k(t)=a_k\cos kt+b_k\sin kt, \quad B_k(t)=b_k\cos kt-a_k\sin kt,$$

and

$$\varphi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\}, \quad \psi(t)=\tfrac{1}{2}\{f(x+t)-f(x-t)\}.$$

The following theorems are proved. (1) If $0<\alpha<1$ and $\int_0^\pi t^{-\alpha}|d\varphi(t)|<\infty$, then $\sum n^\alpha A_n(t)$ is summable $|C, \beta|$ for $\beta>\alpha$, at $t=x$. (2) If $0<\alpha<1$, $\psi(+0)=0$, and

$$\int_0^\pi t^{-\alpha}|d\psi(t)|<\infty,$$

then $\sum n^\alpha B_n(t)$ is summable $|C, \beta|$ for $\beta>\alpha$, at $t=x$.

W. W. Rogosinski (Newcastle-upon-Tyne).

Edmonds, Sheila M. The Parseval formulae for monotonic functions. II. Proc. Cambridge Philos. Soc. 46, 231–248 (1950).

Edmonds, Sheila M. The Parseval formulae for monotonic functions. III. Proc. Cambridge Philos. Soc. 46, 249–267 (1950).

For any $f(t)\in L(0, \infty)$, let $F_c(x)$ and $F_s(x)$ denote the Fourier cosine and sine transforms of f. In the first paper of the sequence [same Proc. 43, 289–306 (1948); these Rev. 8, 577] the author investigated the validity of the Parseval formulas

$$\text{(a)} \qquad \int_0^\infty F_cG_c\,dx=\int_0^\infty fg\,dt, \quad \int_0^\infty F_sG_s\,dx=\int_0^\infty fg\,dt$$

and found that they are usually true when two of the functions appearing in them are monotonic and one of the integrals exists. In the present two notes a similar discussion is given for the Parseval formulas in Fourier series,

$$\text{(b)} \qquad \tfrac{1}{2}a_0\alpha_0+\sum_1^\infty(a_n\alpha_n+b_n\beta_n)=\pi^{-1}\int_0^{2\pi}fg\,dt,$$

$$\text{(c)} \qquad \tfrac{1}{2}a_0'\alpha_0'+\sum_1^\infty a_n'\alpha_n'=2\pi^{-1}\int_0^\pi fg\,dt,$$

$$\text{(d)} \qquad \sum_1^\infty b_n'\beta_n'=2\pi^{-1}\int_0^\pi fg\,dt.$$

Here a_n, b_n are the Fourier coefficients of f; a_n' and b_n' are the coefficients in the half-range cosine and sine series for f; and α_n, β_n, α_n', β_n' are similar quantities for g. Results previously established for (a) remain generally valid for (c) and (d), but now we have a greater variety of theorems since there is less reciprocity in the relations between functions and their Fourier coefficients than in the transform case. It is impossible to enumerate the results obtained and a few that follow may serve as examples. (1) Suppose that $f(t)$ and $g(t)$ are decreasing and in $L(0, 2\pi)$, and that $fg\in L$. Suppose also that $f(t)$ is positive, or, more generally that (†) $f(t)g(2\pi-t)\in L(0, 2\pi)$. Then we have (b). The latter formula may be false if we do not assume (†). (2) If $f(t)$ and $g(t)$ are decreasing and belong to $L(0, \infty)$ and if $fg\in L(0, \infty)$, we have (c) and (d). (3) Suppose that a_n, α_n are decreasing sequences tending to 0 and such that $\sum a_n\alpha_n<\infty$. Let $f(x)=\frac{1}{2}a_0+\sum a_n\cos nx$, $g(x)=\frac{1}{2}\alpha_0+\sum\alpha_n\cos nx$. Then we have $\pi^{-1}\int_{-0}^{2\pi}fg\,dx=\frac{1}{2}a_0\alpha_0+\sum_1^\infty a_n\alpha_n$. *A. Zygmund.*

Bohr, Harald. On limit periodic functions of infinitely many variables. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 145–149 (1950).

After an introductory discussion of purely periodic (p.p.) and limit periodic (l.p.) functions (the latter are defined as uniform limits of p.p. functions), the author proceeds to the treatment of almost periodic (a.p.) functions in denumerably many variables. A function $P(X)=P(x_1, x_2, \cdots)$ is called p.p. with respect to the axis if there is a sequence of real numbers $p_n\neq 0$ such that for each n,

$$P(x_1, x_2, \cdots, x_n+p_n, x_{n+1}, \cdots)=P(X).$$

By closing this set (using uniform convergence) one gets the functions l.p. with respect to the axis. If T is a one-to-one bicontinuous linear transformation of the sequence space over which the $P(X)$ are defined (convergence in this space means coordinate-wise convergence) and if $P(X)$ is p.p. with respect to the axis, the function $P(TX)=P_T(X)$ is called p.p. with respect to T. Let C_T denote the uniform closure of the class $P_T(X)$, and let $\Gamma=\bigcup_T C_T$. The functions of Γ are called the l.p. functions. The main result of the paper is the affirmative answer to a question raised by B. Jessen concerning the (uniform) closedness of Γ. The proof is based upon two successive applications of the following lemma: a necessary and sufficient condition that $\sum A_n e^{iM_n(X)}$ (where $M_n(X)=\sum_{i=1}^{m_n}\lambda_{ni}X_i$), the Fourier expansion of an a.p. function, be the Fourier expansion of an l.p. function is: $\sum_{i=1}^k a_iM_{n_i}=0$ implies that the a_i are rational.

B. Gelbaum (Minneapolis, Minn.).

Guinand, A. P. A note on repeated general transformations. Proc. Cambridge Philos. Soc. 46, 354–355 (1950).

It is shown that if $x^{-1}k(x)$ belongs to $L_2(0, \infty)$,

$$\int_0^\infty k(ax)k(bx)x^{-2}dx=\min(a, b),$$

$f(x, y)$ is homogeneous in x and y of degree -1, $x^{-1}f(1, x)$ belongs to $L(0, \infty)$, then

$$f(y, x) = (\partial^2/\partial x \partial y) \int_0^\infty f(u, v) k(xu) k(yv) u^{-1} v^{-1} du dv.$$

I. I. Hirschman, Jr. (St. Louis, Mo.).

* **Denis-Papin, Maurice, et Kaufmann, A. Cours de calcul opérationnel. (Transformation de Laplace).** Éditions Albin Michel, Paris, 1950. 237 pp. 1200 francs.

This work is designed for students of the engineering sciences. Accordingly, the authors strive for an "exposé logique, simple et réaliste" rather than for mathematical rigour. The operational image of a function is defined by means of the Laplace integral, and all rules of operational calculus are derived from this definition. An indication of the contents follows. Chapter I: Definitions and the fundamental rules of operational calculus, applications to differential, difference, and integral equations, the delta function. Chapter II: Inversion by means of the complex inversion formula, power series, Fourier series, Heaviside's expansion theorem. Chapter III: Impedance operators, Heaviside's unit function. Chapter IV: Notations, symbolic operators, finite Fourier transformation, operational calculus in two variables, and other supplementary remarks. Chapter V: Applications to problems in mechanical and electrical engineering, etc. Chapter VI: Tables of the rules of operational calculus [26 entries], table of transform pairs [202 pairs arranged according to images], a special table of 30 transform pairs arising in transient theory, and a bibliography of 4 pages.

The material is well presented, and is lavishly illustrated by examples. Each chapter is followed by a large number of exercises. The printing is good except for the faulty composition of some fractions and some broken parentheses which make it difficult at times to distinguish between f' and $f($. The level is considerably below Churchill's book [Modern Operational Mathematics in Engineering, McGraw-Hill, New York, 1944; these Rev. **5**, 267] but [from the mathematical point of view] distinctly above some of the other books on operational calculus. *A. Erdélyi.*

Special Functions

Lauwerier, H. A. The asymptotic expansion of the confluent hypergeometric function $M_{\omega,\frac{1}{2},0}(2\omega)$. Nederl. Akad. Wetensch., Proc. **53**, 188–195 = Indagationes Math. **12**, 26–33 (1950).

The asymptotic expansion of the function mentioned in the title for large positive ω has been derived by the author in a forthcoming paper; here he gives a numerical estimate of the remainder in the asymptotic expansion. He starts with an integral representation of the function in question, applies the method of steepest descents, and obtains the remainder in terms of integrals which he proceeds to estimate. A numerical example is given to demonstrate the practical usefulness of the estimate. *A. Erdélyi.*

Toscano, Letterio. I polinomi ipergeometrici nel calcolo delle differenze finite. Boll. Un. Mat. Ital. (3) **4**, 398–409 (1949).

Let Δ be the difference operator with respect to γ, $\Delta f(\gamma) = f(\gamma+1) - f(\gamma)$. If δ and x are independent of γ, we have

$$x^\gamma \Gamma(\gamma+\delta) F(-n, \gamma+\delta; \gamma; x) = (-)^n \Gamma(\gamma) \Delta^n \{x^\gamma \Gamma(\gamma+\delta)/\Gamma(\gamma)\},$$

and there is a similar formula for Laguerre polynomials. With these two formulae as the point of departure, the author uses the operators Δ and $D = d/d\gamma$ to derive a large number of formulae, both old and new, for hypergeometric polynomials, in particular for generalised Laguerre polynomials. *A. Erdélyi* (Pasadena, Calif.).

Toscano, Letterio. Osservazioni su particolari funzioni di Kummer. Boll. Un. Mat. Ital. (3) **4**, 274–278 (1949).

The author considers Kummer's series

$$_1F_1(a; c; x) = 1 + \frac{a}{c}\frac{x}{1} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

when $a = -n$, $c = -kn$, and k, n are positive integers. He expresses the sum of the first $n+1$ terms as a Laguerre polynomial $L_{(kn)}^{(-kn-1)}$, points out that the next $(k-1)n$ terms vanish, and writes the sum of the remaining terms as a $_1F_1$. [Reviewer's remark: The result may be interpreted as an attempt to define the meaningless symbol $_1F_1(-n; -kn; x)$, but it should be noted that it is in contradiction to the more or less natural definition $\lim_{c \to -kn} \lim_{a \to -n} {}_1F_1(a; c; x)$.] If $k = 2$, the Laguerre polynomial appearing in the work can be expressed in terms of Bessel functions. Using the connection between a terminating $_2F_0$ and $_1F_1$, the author transforms a terminating $_4F_1$ into a $_2F_3$, and also expresses a certain product of two Laguerre polynomials as a $_2F_0$.

A. Erdélyi (Pasadena, Calif.).

Meixner, Josef. Reihenentwicklungen von Produkten zweier Mathieuschen Funktionen nach Produkten von Zylinder- und Exponentialfunktionen. Math. Nachr. **3**, 14–19 (1949).

The two-dimensional wave equation $u_{xx} + u_{yy} + (2h/c)^2 u = 0$ can be separated both in elliptic and in polar coordinates. In the equation h is a constant, $2c$ the distance of the two foci of the confocal system, and the pole of the polar coordinates is on the major axis of the confocal system, at distance a from the centre. The situation suggests the possibility of an expansion of a product of Mathieu functions (the normal solutions in elliptic coordinates) in an infinite series of Bessel functions multiplied by exponential functions (the normal solutions in polar coordinates). The author sets up a suitable expansion and by substitution in Mathieu's differential equation he finds that the coefficients must satisfy a five-term recurrence relation. If $a = 0$, he shows that the recurrence relation has only three terms, and that the solution can be expressed, in this particular case, in terms of the coefficients $c_{2r}^{(2)}(h)$ which occur in the well-known expansions of Mathieu functions in terms of exponential functions. He discovers that the coefficients $\sum_{r=-\infty}^{\infty} (-1)^r c_{2r}^{(2)}(h) J_{2r-t}(2ah)$, $t = 0, \pm 1, \pm 2, \cdots$, satisfy the five-term recurrence relations obtained for the general case (J is the Bessel function of the first kind). He proves that with these values of the coefficients his expansion is convergent in a certain region which he specifies, and identifies the solutions of Mathieu's equation and of the associated Mathieu equation whose product is represented by his expansion. *A. Erdélyi.*

Tricomi, Francesco G. Una formula sulla norma della funzione gamma incompleta. Boll. Un. Mat. Ital. (3) **4**, 341–344 (1949).

For the incomplete gamma function $\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt$ the author shows that $\Gamma(\alpha, ix)\Gamma(\alpha, -ix)$ is the Laplace

transform of a linear combination of two hypergeometric functions. This is a generalisation of a formula of Enneper's, and of another formula of a similar type. *A. Erdélyi.*

Tricomi, Francesco. Sulle funzioni di Bessel di ordine e argomento pressochè uguali. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **83**, 3–20 (1949).

The author proves the asymptotic representation

$$\pi J_\nu(\nu+(\nu t/6)^{\frac{1}{3}}) = (6/\nu)^{\frac{1}{3}}A_1(t)-(10\nu)^{-1}[3t^3A_1'(t)+2tA_1(t)]+O(\nu^{-\frac{5}{3}}),$$

where

$$A_1(t)=\tfrac{1}{3}\pi(t/3)^{\frac{1}{3}}\{J_{-\frac{1}{3}}(2(t/3)^{\frac{3}{2}})+J_{\frac{1}{3}}(2(t/3)^{\frac{3}{2}})\}$$

is Airy's function, and a similar asymptotic representation for the Bessel function of the second kind. His method is based on an integral equation of the Volterra type and enables him to find approximations to any desired degree of accuracy. By means of the asymptotic formulas so obtained, he discusses the zeros of Bessel functions of the first and second kind and gives in particular the first three terms of the asymptotic expansion of the first positive zeros of $J_\nu(x)$ and $Y_\nu(x)$ for large ν. *A. Erdélyi* (Pasadena, Calif.).

Delerue, Paul. Note sur les propriétés des fonctions hyperbesséliennes. C. R. Acad. Sci. Paris **230**, 1333–1335 (1950).

The author's summary is as follows. L'objet de cette Note est d'indiquer les méthodes qu'on peut employer dans l'étude des fonctions hyperbesséliennes et les directions dans lesquelles ont été obtenus des résultats. Bien des propriétés connues pour les fonctions de Bessel se généralisent à ces fonctions. *A. Erdélyi* (Pasadena, Calif.).

Barrucand, Pierre-A., et Colombo, Serge. Sur la fonction $\nu(t, n)$. C. R. Acad. Sci. Paris **230**, 1335–1337 (1950).

For the function $\nu(t, n)=\int_n^\infty\{t^s/\Gamma(s+1)\}ds$ the authors derive (by operational methods) three formulas of which the most interesting is $\int_0^\infty[1-e^{-x}\nu(x, 0)]L_n(x)dx=i_{n+1}$ where $L_n(x)$ is Laguerre's polynomial, and the numbers i_{n+1} are defined by the generating function

$$x/\log(1-x)=-1+\sum_{n=1}^{\infty}i_nx^n.$$

A. Erdélyi (Pasadena, Calif.).

Jacobsthal, Ernst. Sur les propriétés d'une classe de séries entières. Norske Vid. Selsk. Forh., Trondheim **21**, no. 6, 22–27 (1948).

The author considers a function defined by H. Wergeland, namely

$$\varphi(z; s)=\sum_{n=1}^{\infty}\frac{n^n}{n!}\frac{z^n}{n^s}.$$

He proves the following theorem. If s is an integer $\varphi(z; s)$ is a rational function of $\varphi(z; 1)$. For s positive this function is a polynomial of degree s. For $s=0, -1, -2, \cdots$, $\varphi(z; s)$ is a rational fractional function of $\varphi(z; 1)$ but a polynomial in $\varphi(z; 0)$; the degree of this polynomial is $2|s|+1$.

T. Fort (Athens, Ga.).

Fettis, Henry E. A note on the evaluation of a definite integral. J. Aeronaut. Sci. **17**, 184–185 (1950).

The integral $F(iz)=\int_0^\infty(zt)^{-\frac{1}{2}}e^{-it}[z+t-(z^2+t^2)^{\frac{1}{2}}]dt$ is evaluated in terms of Bessel functions, Lommel-Weber functions, and an indefinite integral over a Lommel-Weber function.

A. Erdélyi (Pasadena, Calif.).

Differential Equations

Schäfke, Friedrich Wilhelm. Zur Parameterabhängigkeit beim Anfangswertproblem für gewöhnliche lineare Differentialgleichungen. Math. Nachr. **3**, 20–39 (1949).

For

$$F[y]\equiv y^{(n)}(x)+f_1(x)y^{(n-1)}(x)+\cdots+f_n(x)y(x),$$
$$G_\kappa[y]\equiv g_{\kappa 0}(x)y^{(m_\kappa)}(x)+\cdots+g_{\kappa m_\kappa}(x)y(x),$$

$m_\kappa<n$; $\kappa=1, \cdots, k$, and $y=y(x, \lambda)$ the solution of

$$F[y]=\lambda G_1[y]+\lambda^2G_2[y]+\cdots+\lambda^kG_k[y]$$

satisfying $y^{(\nu)}(x_0, \lambda)=a_\nu$ ($\nu=0, 1, \cdots, n-1$), the author establishes various results on the order and type of $y^{(\nu)}(x, \lambda)$ ($\nu=0, 1, \cdots, n-1$), as entire functions of the parameter λ; special attention is given to the case $k=1$. These results are then applied to the study of the existence and distribution of proper values for certain types of boundary value problems. *W. T. Reid* (Evanston, Ill.).

Hartman, Philip, and Wintner, Aurel. On non-linear differential equations of first order. Amer. J. Math. **72**, 347–358 (1950).

Two problems are considered for the equation $dy/dx=f(x, y)$ ($x\geqq 0$, $y\geqq 0$). (A) Are solutions completely monotone? A specimen result, for $F+Gy'=0$ with F, G positive and of class C^∞, is: yes, if $(-1)^m\partial^{m+n}F/\partial x^m\partial y^n\geqq 0$ and $(-1)^{m+1}\partial^{m+n}G/\partial x^m\partial y^n\geqq 0$ for $m+n>0$. (B) Are there any "unrestricted" solutions (existing on an interval $x_0\leqq x<\infty$)? Results are obtained partly by comparing with a Riccati equation, partly by using previous results of Wintner [same J. **68**, 553–559 (1946); these Rev. **8**, 272]. The results for (A) and (B) can be combined to obtain criteria for the existence of solutions $y(x)$ of $y'=f(x, y)$ representable for $x\geqq x_0$ as Laplace-Stieltjes integrals $y(x)=\int_0^\infty e^{-xt}d\phi(t)$, with ϕ increasing. *G. E. H. Reuter* (Manchester).

Ważewski, T. Sur l'allure asymptotique des intégrales d'une équation différentielle non linéaire. Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Ser. A. Sci. Math. **1949**, 62–66 (1949).

In the differential equation $y'=f(x, y)$, let $f(x, y)$ be a continuous function on the partly open rectangle $R: 0<x\leqq a$, $|y|\leqq b$ and let $y=y(x)$ be a solution for $0<x\leqq a'(\leqq a)$. It is shown that $y(+0)$ exists and is 0 when the following holds: (i) if $0<|y_0|\leqq b$, then $f(x, y)\neq 0$ for (x, y) in R and sufficiently near $(0, y_0)$ and (ii) no solution $x=x(y)$ in R of $dx/dy=1/f(x, y)$ is such that either $x(y_0-0)$ or $x(y_0+0)$ exists and is 0. The proof depends on geometrical arguments similar to those used by Hartman and Wintner [Amer. J. Math. **68**, 301–308 (1946); these Rev. **7**, 444]. The above result allows a generalization of the main theorem of Hartman and Wintner. [In theorem 3, the misprint "$xf(x, y)$" for "$yf(x, y)$" occurs several times.] *P. Hartman.*

Ważewski, T. Sur certains lemmes relatifs au prolongement des intégrales des équations différentielles ordinaires. Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1949**, 73–74 (1949).

This paper gives three lemmas which imply the following theorem. If (i) through each point (x_0, y_0) of an open set Ω there passes a solution of the differential equation $y'=f(x, y)$; (ii) $f(x, y)$ is continuous in Ω; (iii) $y(x)$ is defined in a neighborhood of x_0; (iv) $y(x)$ belongs to a normed vector Banach space, then $y(x)$ can be prolonged to the boundary of Ω. *J. M. Thomas* (Durham, N. C.).

Haag, Jules. Sur certains systèmes différentiels à solution périodique lentement variable. C. R. Acad. Sci. Paris **230**, 1229–1231 (1950).

In earlier papers [e.g., Bull. Sci. Math. (2) **71**, 205–219 (1947); these Rev. **10**, 194], the author has studied the asymptotic behavior of solutions of the system of differential equations $dx/dt=\lambda f(x,t)$, where $f(x,t)$ is periodic in t and subject to certain general conditions (too lengthy to state here) and λ is a small parameter. In the present note, he discusses the asymptotic behavior of the solutions of $dx/dt=\lambda f(x,t,t)$, where for a fixed θ, $f(x,t,\theta)$ is subject to restrictions similar to those above on $f(x,t)$ and, in addition, $\partial f/\partial\theta$ exists, is small, and satisfies a Lipschitz condition with a small Lipschitz constant. *P. Hartman.*

Biernacki, M. Sur un théorème dans la théorie des équations différentielles. Prace Mat.-Fiz. **47**, 129–141 (1949).

This paper establishes a comparison theorem which asserts that $x_j(t)\leqq y_j(t)$ for $j=0,1,\cdots,n-i$ under the following conditions: (i) $x_j(t_0)=y_j(t_0)$ for $j=0,\cdots,n-i-1$; (ii) $x_j(t_0)\leqq y_j(t_0)$ for $j=n-i,\cdots,n-1$;

$$\text{(iii)}\qquad x'_{n-1}+\sum_{j=0}^{n-1}b_j(t)x_j\leqq 0\leqq y'_{n-1}+\sum_{j=0}^{n-1}a_j(t)y_j;$$

(iv) the index values $0,\cdots,n-1$ can be separated into two mutually exclusive and exhaustive sets so that $a_p-b_p\leqq 0\leqq x_p$, $x_q\leqq 0\leqq a_q-b_q$ for p, q in the respective sets; (v) x_j, y_j for $j=0,\cdots,n-2$ satisfy certain nonhomogeneous linear differential equations of the first order whose coefficients together with the a's and b's are continuous and satisfy certain comparison inequalities on the interval (t_0,T) considered. *J. M. Thomas* (Durham, N. C.).

Mitrinovitch, Dragoslav S. Sur un cas de réductibilité d'équations différentielles linéaires. C. R. Acad. Sci. Paris **230**, 1130–1132 (1950).

The equation (1) $\varphi_0(x)y^{(n)}+\varphi_1(x)y^{(n-1)}+\cdots+\varphi_n(x)y=0$ is said to be reducible if there exists a system of linear equations

$$\text{(2)}\qquad f_{k1}y'_{n-1}+f_{k2}y_{k-1}=y_k,\quad f_{n1}y'_{n-1}+f_{n2}y_{n-1}=0,\quad y_0=y;\ k=1,\cdots,n-1,\ f_{11}\neq 0,$$

such that (1) results from (2) by the elimination of $y_1,\cdots,y_{n-1}$. The author writes out various conditions on the constants a, b, A, B, C which insure, in each of many cases, that the equation $y''+(ax+b)y'+(Ax^2+Bx+C)y=0$ is reducible. Proofs of more general results along this line are to appear in a later paper. *F. G. Dressel.*

* **Moran, Charles William. Asymptotic Theory of Linear Differential Equations Singular in Several Parameters.** Abstract of a Thesis, University of Illinois, 1940. ii+13 pp.

This thesis abstract is concerned with the ordinary linear differential equation

$$\sum_{k=0}^{n}(\lambda_1\lambda_2\cdots\lambda_m)^{(n-k)H}a_{n-k}(x,\lambda_1,\cdots,\lambda_m)y^{[k]}(x,\lambda_1,\cdots,\lambda_m)=0,$$

of the nth order, depending upon m large parameters $\lambda_1,\lambda_2,\cdots,\lambda_m$. In terms of these the coefficients are of the form

$$a_{n-k}(x,\lambda_1,\cdots,\lambda_m)=\sum_{\mu_1=0}^{\infty}\cdots\sum_{\mu_m=0}^{\infty}a_{n-k,\mu_2,\cdots,\mu_m}(x)\lambda_1^{-\mu_1}\lambda_2^{-\mu_2}\cdots\lambda_m^{-\mu_m};$$

H is a positive integer or zero, and the $a_{i,\mu_2,\cdots,\mu_m}(x)$ are indefinitely differentiable, with $a_0=1$ and $a_n\neq 0$. The characteristic equation

$$\sum_{k=0}^{n}a_{n-k}(x,\infty,\cdots,\infty)\rho^k=0$$

is assumed to have distinct (nonvanishing) roots. The equation admits of formal solutions

$$S_i(x,\lambda_1,\cdots,\lambda_m)=\exp\{(\lambda_1\lambda_2\cdots\lambda_m)^H Q_i\}\cdot{}_i\sigma(x,\lambda_1,\cdots,\lambda_m),$$

in which

$$Q_i=\sum_{\mu_1=0}^{H-1}\cdots\sum_{\mu_m=0}^{H-1}{}_iq_{\mu_2,\cdots,\mu_m}(x)\lambda_1^{-\mu_1}\lambda_2^{-\mu_2}\cdots\lambda_m^{-\mu_m},$$

$${}_i\sigma=\sum_{\mu_1=0}^{\infty}\cdots\sum_{\mu_m=0}^{\infty}{}_i\sigma_{\mu_2,\cdots,\mu_m}(x)\lambda_1^{-\mu_1}\lambda_2^{-\mu_2}\cdots\lambda_m^{-\mu_m},$$

and the theorem is to the effect that for x and $\lambda_1,\lambda_2,\cdots,\lambda_m$, respectively on an interval and in a region for which the real parts of the expressions $(\lambda_1\lambda_2\cdots\lambda_m)^H Q_i$ retain an ordering as to magnitude there is a full set of actual solutions that are represented by the $S_i(x,\lambda_1,\cdots,\lambda_m)$ to terms of specified degrees in the $1/\lambda_i$. *R. E. Langer.*

Turrittin, H. L. Stokes multipliers for asymptotic solutions of a certain differential equation. Trans. Amer. Math. Soc. **68**, 304–329 (1950).

The differential equation $y^{(n)}-x^\nu y=0$ ($n\geqq 2$, ν a positive integer) possesses n linearly independent solutions $y_j(x)$ ($j=1,\cdots,n$), such that $x^{1-j}y_j(x)$ is an integral function of x^p ($p=n+\nu$), which takes a nonzero value at $x=0$. From a general theorem of Trjitzinsky [Acta Math. **62**, 167–226 (1933)] it follows that in each sector $\pi h/p\leqq\arg x\leqq\pi(h+1)/p$ ($h=0,1,\cdots,2p-1$) there are also n linearly independent solutions $\bar{y}_k(x)$ such that $e^{-Y_k}Y_k^{t-1}\bar{y}_k(x)$ may be expanded asymptotically in ascending powers of Y_k^{-1}; here $t=\frac{1}{2}+\frac{1}{2}n-\frac{1}{2}p^{-1}n(n-1)$ and $Y_k=(n/p)x^{p/n}\exp(2\pi ik/n)$. In each of these sectors $y_k(x)$ may be written as a linear form in $\bar{y}_1(x),\cdots,\bar{y}_n(x)$, the coefficients of which are called the Stokes multipliers. The author evaluates these multipliers by means of the Wright-Ford-Newsom-Hughes method, to which he adds a theorem on the behaviour of

$$e^z\int_{a-i\infty}^{a+i\infty}z^{-s}\Gamma(s)P(s)ds$$

for $|z|\to\infty$ if $|\arg z|<\pi/2$. The author carries out his investigation not only for the specified sectors, but also for sectors which are considerably larger and which are chosen so that the Stokes multipliers are determined uniquely.

J. G. van der Corput (Stanford University, Calif.).

Olsson, Herbert. On certain asymptotic solutions of Riccati's differential equation. Ark. Mat. **1**, 21–25 (1949).

This paper is concerned with the equation

$$y''-[\rho^2f^2(x)+g(x)]y=0$$

with ρ large and positive; $a\leqq x\leqq b$ and $f^2(x)$ bounded from zero. Its purpose is to show that there is a solution of the form

$$y=\exp\left\{\pm\rho\int_a^x f(x)dx\right\}\cdot\{P_{n-1}(x,1/\rho)+O(\rho^{-n})\},$$

with P_{n-1} a polynomial of degree $(n-1)$ in $1/\rho$. Neither the result nor the method, which depends upon transformation

of the equation into a Riccati form, is new. [In his introduction the author attributes to the reviewer aspersions upon the so-called W.K.B. method which the reviewer herewith disclaims. The misinterpretation is attributable, no doubt, to linguistic difficulties.] *R. E. Langer.*

Koppe, Heinz. Die Reflexion einer Welle an einer Potentialschwelle. Z. Naturforschung **5a**, 137–139 (1950).

The wave equation $\psi''+p^2(x)\psi=0$ is considered for cases in which $p^2(x)$ is positive and becomes essentially constant for large positive and large negative x. If the solution which is effectively e^{-ipx} near $x=-\infty$ has the form $C_+e^{ipx}+C_-e^{-ipx}$ near $x=+\infty$, the coefficient of reflection of the medium characterized by $p^2(x)$ is $R=|C_+/C_-|^2$. The paper gives an approximate formula for R when the reflection is small. The method is to transform the equation by the substitution $\psi'=[-ip(x)(1-v)/(1+v)]\psi$ into the Riccati equation $v'-2ipv=(p'/2p)(1-v^2)$. If $v_1(x)$ is such that $v_1(-\infty)=0$ then $R=\lim_{x\to\infty}|v_1|^2$. Assuming that v_1^2 is negligibly small the Riccati equation is replaceable by a linear equation, and from this the result

$$R\approx\tfrac{1}{4}\left|\int_{-\infty}^{\infty}\exp\left[-2i\int p\,dx\right]d\log p(x)\right|^2$$

is drawn. *R. E. Langer* (Madison, Wis.).

Cherry, T. M. Uniform asymptotic formulae for functions with transition points. Trans. Amer. Math. Soc. **68**, 224–257 (1950).

The author considers the asymptotic expansions for large $|\nu|$ of the solutions of a differential equation

$$d^2y/dz^2+y\{-\nu^2z+g(z,\nu^{-2})\}=0, \tag{1}$$

under the following conditions. (1) In a domain D_z of the z-plane, enclosing $z=0$, the function g has an expansion $g(z,\nu^{-2})=\sum_0^\infty\nu^{-2r}g_r(z)$, where the g_r are regular and $|g_r(z)|\leqq Mk^r$ for $|z|\leqq 1$, $|g_r(z)|\leqq Mk^r|z|$ for $|z|\geqq 1$. (2) The domain D_z is a star-domain relative to $z=0$; it contains sectors of nonzero angles, extending to infinity, centered on the rays $\arg z=0$, $\pm 2\pi/3$. (3) In any sector of D_z which extends to infinity, $g(z,\nu^{-2})=\alpha_\nu z+\beta_\nu+O(z^{-1})$; $\alpha_\nu=\sum_0^\infty\alpha_r\nu^{-2r}$; $\beta_\nu=\sum_0^\infty\beta_r\nu^{-2r}$, uniformly for $\arg z$ belonging to the sector and $|\nu|$ sufficiently large; the series for α_ν and β_ν being convergent.

The asymptotic expressions are obtained by comparing with the Airy equation $d^2\eta/d\zeta^2-\nu^2\zeta\eta=0$, which is transformed into an equation

$$d^2y/dz^2+y(-\nu^2z+g-G)=0 \tag{2}$$

by a transformation $\zeta=\varphi(z)=z+\sum_1^\infty\nu^{-2r}\varphi_r(z)$ and $\eta=\varphi^{\frac{1}{2}}y$. The functions φ_r can be calculated successively from a set of differential equations so as to have the function G satisfy in D_z the inequalities $|G|\leqq A_n|\nu|^{-2n}$ for $|z|\leqq 1$, $|G|\leqq A_n|\nu|^{-2n}|z|^{-7/2}$ for $|z|\geqq 1$. In order to prove the approximate equality of corresponding solutions of (1) and (2), equation (2) is transformed into the Bessel form

$$d^2w/dv^2+w(-\nu^2+5/(36v^2))=0 \tag{3}$$

by putting $\zeta=(3v/2)^{\frac{2}{3}}$, $\eta=w\zeta^{-\frac{1}{2}}$ and (1) into a form

$$d^2\tilde{\omega}/dv^2+\tilde{\omega}\{-\nu^2+5/(36v^2)+F(v,\nu^{-2})\}=0, \tag{4}$$

where $F=G\varphi'^{-2}(\frac{2}{3}v^{-1})^{\frac{1}{2}}$. The comparison of the solutions of (3) and (4) rests upon the transformation of (4) into an integral equation, using the known solutions of (3).

Taking as a fundamental pair of solutions the functions $L(\nu v)=(\nu v)^{\frac{1}{2}}K_{\frac{1}{3}}(\nu v)$ and $L(\nu ve^{-\pi i})$, the integral equation has the form

$$\frac{\tilde{\omega}(v)}{L(\nu v)}=1+\frac{1}{\nu\pi}\int_v^\infty\left\{L(\nu t)L(\nu te^{-\pi i})-L^2(\nu t)\frac{L(\nu ve^{-\pi i})}{L(\nu v)}\right\}\frac{F(t)\tilde{\omega}(t)}{L(\nu t)}dt.$$

Taking ν real and positive and v and the path of integration in a region I, defined by

$$-\tfrac{2}{3}\pi+\epsilon\leqq\arg v\leqq\tfrac{2}{3}\pi-\epsilon;\quad -\tfrac{2}{3}\pi+\epsilon\leqq\arg t\leqq\tfrac{2}{3}\pi-\epsilon; \tag{5}$$

$$-\pi/2\leqq\arg(t-v)\leqq\pi/2, \tag{6}$$

the iteration process used in solving the integral equation is shown to be convergent for sufficiently large $|\nu|$; (6) excludes certain "shadow" zones, which cannot be "seen" from $v=-i\infty$. Then, uniformly in region I,

$$\tilde{\omega}_1(v)/L(\nu v)=1+O(\nu^{-2n-1}).$$

Similarly in a region II, defined by $-\pi/2+\epsilon\leqq\arg v\leqq\pi/2-\epsilon$, a relation

$$\tilde{\omega}_2(v)/L(\nu ve^{-\pi i})=1+O(\nu^{-2n-1})$$

is obtained, and in a region III, the conjugate of II, the relation $\tilde{\omega}_3(\nu)/L(\nu ve^{-\pi i})=1+O(\nu^{-2n-1})$ holds.

As the functions L can easily be expressed by means of Airy functions of respective arguments $\nu^{\frac{2}{3}}\zeta$, $\nu^{\frac{2}{3}}\omega^2\zeta$, $\nu^{\frac{2}{3}}\omega\zeta$, where $\omega=\exp 2\pi i/3$, asymptotic expressions for certain "standard solutions" $Y_{\nu 1}$, $Y_{\nu 2}$ and $Y_{\nu 3}$ of equation (1) are found. By using linear relations between these standard solutions, whose coefficients can be determined up to order ν^{-2n-1}, the author succeeds in extending the domain of validity of the asymptotic expressions to other sectors, such as, e.g., $|\arg(-\zeta)|\leqq 2\pi/3$ in region I. In the neighbourhood of a zero of the Airy function which lies in this sector, a small modification is necessary. Finally it is possible to extend the validity of the formulae to the case $|\arg\nu|\leqq\pi/2$.

The solutions of the differential equation are expressed by means of Airy functions whose arguments are represented by an asymptotic series; by expanding the Airy function into a Taylor series, the author obtains asymptotic expansions for these solutions. The general formulae are applied to two particular cases: the Bessel functions of large order and the Chaplygin hypergeometric functions occurring in the theory of compressible flow. In the latter case instead of using the Airy functions directly it is found more useful to approximate the hypergeometric functions by the Bessel functions of large order which occur in the first example.

J. G. van der Corput (Stanford University, Calif.).

Titchmarsh, E. C. Some theorems on perturbation theory. Proc. Roy. Soc. London. Ser. A. **200**, 34–46 (1949).

This paper is mainly concerned with the equation

$$d^2\varphi/dx^2+[\lambda-q(x)-\epsilon\sigma(x)]\varphi=0$$

on the interval $0\leqq x<\infty$, with the conditions

$$[\varphi\cos\alpha+\varphi'\sin\alpha]_{x=0}=0,$$

$\int_0^\infty\varphi^2dx$ convergent. Here ϵ is a small positive parameter, and $q(x)$ and $\sigma(x)$ are taken to be continuous and such that for each ϵ the characteristic values are a discrete set $\lambda_n(\epsilon)$ having ∞ as its only limit point. For the analysis of the perturbed system (with $\epsilon>0$) in comparison with the unperturbed ($\epsilon=0$) it is frequently assumed that the values $\lambda_n(\epsilon)$ and the associated characteristic solutions $\varphi_n(x,\epsilon)$ are expressible as power series in ϵ, the coefficients of which are determinable by substitution in the differential equation. The author eschews such an assumption, and shows how

under appropriate hypotheses upon $q(x)$ and $\sigma(x)$ relations

$$\varphi_n(x,\epsilon)=\varphi_n(x,0)+\epsilon\sum_{\nu\neq n}\frac{a_{n,\nu}\varphi_\nu(x,0)}{\lambda_n(0)-\lambda_\nu(0)}+O(\epsilon^2),$$

$$\lambda_n(\epsilon)=\lambda_n(0)+\epsilon a_{n,n}+\epsilon^2\sum_{\nu\neq n}\frac{a^2_{n,\nu}}{\lambda_n(0)-\lambda_\nu(0)}+O(\epsilon^3)$$

with $a_{n,m}=\int_0^\infty\sigma(x)\varphi_n(x,0)\varphi_m(x,0)dx$ may be deduced. The extension of the results to the equation $\nabla^2\varphi+(\lambda-q)\varphi=0$ is indicated.

R. E. Langer (Madison, Wis.).

Rubinowicz, A. Eigenfunctions following from Sommerfeld's polynomial method. Proc. Phys. Soc. Sect. A. **62**, 736–738 (1949).

In an earlier paper [Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **40** (1947), 57–63 (1948); these Rev. **11**, 517] the author has considered the conditions under which the Sturm-Liouville equation may be solved by the form $E(x)P(x)$ where P is a finite polynomial in x and satisfies

$$x^2(A_2+B_2x^k)P''+2x(A_1+B_1x^k)P'+(A_0+B_0x^k)P=0.$$

The solution P of this equation is given in terms of the hypergeometric function ($B_2\neq0$) and in terms of the confluent hypergeometric function ($B_2=0$). *H. Feshbach.*

Bulgakov, B. V., and Kuzovkov, N. T. On the accumulation of disturbances in linear systems with varying parameters. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 7–12 (1950). (Russian)

Consider the vector-matrix system $dz/dt=F(t)z+w(t)$, $z(0)=c$, a constant vector, where z and w are column vectors and $F(t)$ is a matrix. The problem is that of determining a bound on z in terms of the bounds on $w(t)$ and the solution of the homogeneous equation $dy/dt=F(t)y$. Let $Y(t)$ denote the solution of $dY/dt=F(t)Y$, $Y(0)=I$. Then

$$z=y+\int_0^t Y(t)\,Y^{-1}(t_1)w(t_1)dt_1,$$

whence

$$\|z\|\leqq\|y\|+\int_0^t\|Y(t)\,Y^{-1}(t_1)\|\,\|w(t_1)\|dt_1,$$

where we set $\|y\|=\sum_i|y_i|$, $\|A\|=\sum_{i,j}|a_{ij}|$, if y has the components y_i, A the elements a_{ij}. The difficulty lies in estimating $\|Y(t)\,Y^{-1}(t_1)\|$. This can be done, with a certain amount of numerical calculation, if $F(t)$ is a constant matrix, since $Y(t)\,Y^{-1}(t_1)=Y(t-t_1)$ in this case. Consequently, the authors divide the t-range into intervals $[0,t_1]$, $[t_1,t_2]$, and so on, and take $F(t)$ constant and equal to $F(t_i)$ in the ith interval. Numerical examples are given. *R. Bellman.*

Butenin, N. V. On the theory of "resonance" in a mechanical auto-oscillating system with gyroscopic terms. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 45–56 (1950). (Russian)

In an earlier paper [same journal **13**, 337–348 (1949); these Rev. **11**, 249] the author discussed a system of the fourth order with forced oscillation $\mu Q\sin t$ reducing to a linear system for $\mu=0$. There were two fundamental frequencies ω_1, ω_2 and only the nonresonance case where both were far from unity was considered. The resonance case is taken up in the present paper with application here also to a monorail system with gyroscopic stabilizer.

S. Lefschetz (Princeton, N. J.).

Ryabov, B. A. Determination of the parameters of the régime of established auto-oscillation of certain systems. Doklady Akad. Nauk SSSR (N.S.) **71**, 663–666 (1950). (Russian)

In connection with the oscillation of a certain system, the following system of equations is encountered:

$$\sum_{k=1}^{n}P_{ki}(p)x_k=\delta A_i\sigma_0,$$

where $\delta=1$ for $x_\sigma\geqq x_{\sigma0}$, $\dot{x}_\sigma\geqq0$, $\delta=-1$ for $x_\sigma\leqq-x_{\sigma0}$, $\dot{x}_\sigma\leqq0$. Here $p=d/dt$, $P_{kj}=a_{kj}p^2+b_{kj}p+c_{kj}$; a_{kj}, b_{kj}, c_{kj}, A_j and σ_0 are constants. The author shows how to solve this system in terms of the solution of the linear homogeneous equation.

R. Bellman (Stanford University, Calif.).

Bautin, N. N. Theory of an escapement regulator with spring blade. Doklady Akad. Nauk SSSR (N.S.) **72**, 19–22 (1950). (Russian)

In various mechanisms, among them those used to measure very short time intervals, there occur systems such as here considered. The basic differential equations between consecutive impulses are $M\ddot{y}=P$, $m\ddot{x}=-\kappa_0x-b\dot{x}$, where the coefficients are all constant. Each impulse yields a linear transformation Ω on the values of x, $\dot{x}$, $\dot{y}$ "before" and "after," where Ω depends on various parameters of the system. The fixed points of Ω determine the periodic regimes. Their determination is reduced to the solution of a certain transcendental equation. In the absence of friction the equation becomes a quadratic whose unique positive root determines the periodic regime and it may be stable or unstable. When there is friction there may be present many periodic regimes, some stable, some unstable. As many as four stable periodic regimes have actually been verified experimentally with simple models.

S. Lefschetz (Princeton, N. J.).

Železcov, N. A. Vyšnegradskiĭ diagrams for an isodromic regulator with nonlinear action. Avtomatika i Telemehanika **10**, 424–436 (1949). (Russian)

A Vyšnegradskiĭ diagram for a mechanism is a chart showing for what values of the parameters of the mechanism the linearized equations governing the device have only stable solutions. The paper is a detailed discussion of the stability of a particular regulating system.

E. N. Gilbert (Murray Hill, N. J.).

***Finikov, S. P. Metod vnešnih form Kartana v differencial'noĭ geometrii. Teoriya sovmestnosti sistem differencial'nyh uravneniĭ polnyh differencialah i v častnyh proizvodnyh. [Cartan's Method of Exterior Forms in Differential Geometry. The Theory of Compatibility of Systems of Total and Partial Differential Equations.]** OGIZ, Moscow-Leningrad, 1948. 432 pp.

The subject matter of this book is considerably broader than the title would indicate. The first chapter gives a thorough treatment of the theory of systems of partial differential equations starting with the work of Cauchy and Kovalevskaia and ending with that of Riquier and J. M. Thomas. The theory of exterior forms is approached through differential geometry, but the algebraic operations are axiomatized (the Grassmann ring) and illustrated by many examples. After defining exterior differentiation the author

derives the usual differential and integral theorems. The subject of Pfaffian systems is then taken up in great detail [four chapters], the proofs being uniformly clear and precise. This is followed by a thorough treatment of general exterior differential equations, a subject usually not found in textbooks. It is based primarily on the work of Cartan, Kähler and Thomas. There are a number of problems ranging from fairly simple examples to important theorems. The last chapter deals with applications of exterior differentiation to infinitesimal Lie groups and to differential geometry based on the "repère mobile" of Cartan.

M. S. Knebelman (Pullman, Wash.).

*Vekua, I. N. **Novye metody rešeniya èlliptičeskih uravnenil. [New Methods for Solving Elliptic Equations].** OGIZ, Moscow-Leningrad, 1948. 296 pp.

General representation of the solutions of linear differential equations of the second order of elliptic type with two independent variables. Transformation and approximate solution of equation (E_0) $[\Delta u+au_x+bu_y+cu=0]$. Boundary problems. General representation of the solution of a system of differential equations of the second order of elliptic type and its applications. General representation of the solutions of a class of differential equations of elliptic type of higher order and its applications. Applications in the theory of elasticity. (Plane problem of stationary oscillations of an elastic cylinder. Bending of thin plates. Application to the theory of a thin spherical shell. Application to the theory of thin sloping elastic shells.)

Table of contents.

Castelluccio, Domenico. Nuovo metodo di analisi dei fenomeni di propagazione per onde. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **11(80)** (1947), 281–328 (1949).

The author solves a hyperbolic system of 2 first order partial differential equations in 2 independent variables by a method which appears to be equivalent to Riemann's method of integration.

F. John (New York, N. Y.).

Landau, H. G. A problem in radiobiology: Diffusion and recombination of ions. Bull. Math. Biophys. **12**, 27–34 (1950).

The concentration of either positive or negative ions in a homogeneous medium of infinite extent after that medium has been penetrated by a single ionizing particle or ray satisfies a nonlinear partial differential equation of the type $N_t=N_{rr}+N_r/r-bN^2$, where $N(r, t)$, r, and t are proportional to the concentration, distance from the ray, and time, respectively, and b is a prescribed constant. The equation takes account of the diffusion of ions outward from the ray together with the recombination of the ions at a rate proportional to the product of the concentrations of positive and negative ions. An initial condition of the type $N(0, t)=\exp(-r^2)$ is assumed. A perturbation method is used to find an approximate solution for $N(r, t)$ using the first three steps of the method. From that solution a formula for the total number of ions in the medium at any time, per unit length of ray, is obtained in a form that is suitable for computation. The case in which the coefficient of diffusion is different for the two types of ions is also considered.

R. V. Churchill (Ann Arbor, Mich.).

Difference Equations, Special Functional Equations

Arrighi, Gino. Sulla equazione funzionale

$$2\varphi(x)\varphi(y)=\varphi(x+y)+\varphi(x-y).$$

Boll. Un. Mat. Ital. (3) **4**, 255–257 (1949).

E. Picard [Leçons sur quelques équations fonctionelles . . . , Gauthier-Villars, Paris, 1928, chap. 1] has proved that if there exists a point where $\varphi(x)$ is finite and does not vanish then if $\varphi(x)$ is continuous the functional equation mentioned in the title has $\cos \lambda x$ and $\cosh \lambda x$ as its only continuous solutions. The author shows that the condition of continuity can be replaced by the weaker condition of the existence of a limit on one side. *J. Aczél* (Miskolc).

Biernacki, Mieczysław. Sur l'équation $\frac{\Delta^2 y}{h^2}+A(x)y=0$.

Prace Mat.-Fiz. **47**, 49–60 (1949).

The author studies the equation (1) $\Delta^2 y+h^2A(x)y=0$, where h is a positive constant and $A(x)$ is a continuous function satisfying the inequality $A(x)\geqq m>0$ on a given interval. He establishes a number of comparison and oscillation theorems for this equation analogous to those encountered in the Sturmian theory of differential equations. Some of the results obtained may be stated briefly as follows. (a) Every continuous solution of the equation (1) in which $A(x)$ satisfies the stated conditions for $x>x_0$ is oscillatory for $x>x_0$. (b) Let $y(x)$ be a solution of (1) such that $y(\alpha)\geqq 0$ and $y(x)>0$, $\alpha<x\leqq\beta$. Let n be the positive integer such that $nh\leqq\beta-\alpha<(n+1)h$. Let $z(x)$ satisfy $\Delta^2 z+h^2a(x)z=0$ where $A(x)\geqq a(x)>0$, $z(\beta)=y(\beta)$, and $z(\beta-h)=y(\beta-h)$. Then $z(\beta-ih)\geqq y(\beta-ih)$, where $i=2, 3, \cdots, n$. (c) Let (1) be satisfied on an interval I. Then every sub-interval of I having length greater than $\pi m^{-\frac{1}{2}}+6h$ contains at least one zero of every solution of (1) continuous on I. Further, if a and b are consecutive zeros of a given continuous solution and c is the point at which $|y(x)|$ attains its maximum on $a\leqq x\leqq b$, then $c-a<\frac{1}{2}(\pi m^{-\frac{1}{2}}+7h)$, $b-c<\frac{1}{2}(\pi m^{-\frac{1}{2}}+5h)$.

P. E. Guenther (Cleveland, Ohio).

Integral Equations

Lewis, D. C. Comments on the classical theory of integral equations. J. Washington Acad. Sci. **40**, 65–71 (1950).

The author gives an exposition of the theory of singular functions and singular values of a nonsymmetric continuous kernel, differing from E. Schmidt's original account [Math. Ann. **63**, 433–476 (1907)] by making no use of the fact that the pairs of singular functions of $k(s, t)$ are the characteristic functions of $\int k(s, u)k(t, u)du$ and $\int k(u, s)k(u, t)du$. He gives a direct proof of the existence of nontrivial pairs of singular functions for an arbitrary continuous kernel, and deduces the standard expansion theorem; he also proves an extremal property of the singular values, analogous to the well-known property of the characteristic values of a symmetric kernel. [The author's names for a pair of singular functions and a singular value are "characteristic pair" and "characteristic number."] *F. Smithies* (Cambridge, England).

Umeda, Kwai. Zur Blochschen Integralgleichung. Sci. Papers Inst. Phys. Chem. Res. **39**, 342–347 (1942).

An integral equation of the form

$$\chi(z)=\varphi(z)-\int_{z-a}^{z+a}K(z, \zeta)\varphi(\zeta)d\zeta$$

where K is symmetric is considered. It is shown that the linear equations obtained by substituting a power series with undetermined coefficients for $\varphi(\zeta)$ may also be obtained by employing this series as a trial function in the variation principle for φ. *H. Feshbach* (Cambridge, Mass.).

Kato, Tosio. On the convergence of the perturbation method. I. Progress Theoret. Physics 4, 514–523 (1949).

The author considers the eigenvalue problem $H_\kappa f_\kappa = \lambda_\kappa f_\kappa$, $H_\kappa = H_0 + \kappa V$. A regular perturbation is defined in terms of the resolvent $R_\kappa = (H_\kappa - l)^{-1}$, as one for which R is regular in κ for some fixed l. It is then shown that for such perturbations a power series in κ exists for λ_κ and f_κ. The convergence is shown to depend essentially on the operator VR_0. If now the distance between eigenvalues is d, then the series for H_κ in terms of VR_0 converges if $\|\kappa V\| \leqq d/2$.

H. Feshbach (Cambridge, Mass.).

Lehmann, N. Joachim. Beiträge zur numerischen Lösung linearer Eigenwertprobleme. I. Z. Angew. Math. Mech. 29, 341–356 (1949). (German. English, French and Russian summaries)

The author develops a method of obtaining upper and lower bounds enclosing the ith eigenvalue of the integral equation (1) $y(x) = \lambda \int K(x, s) y(s) ds$. The method consists of a modification of a maximum-minimum principle of Courant in which the $1/\lambda_i$ occur as the extrema of the integral form (2) $\int\int K(x, s) f(x) f(s) dx ds$ under varying orthogonality restrictions on f, so that approximate maximisation (minimisation) of (2) yields lower (upper) bounds for the $1/\lambda_i$. The author's modification is based on the solution of the Fredholm equation $f(x) = y(x) - t_i \int K(x, s) y(s) ds$ in the form $y(x) = f(x) + t_i \int R(x, s; t_i) f(s) ds$ and (2) is replaced by (3) $\int\int R(x, s; t_i) f(x) f(s) dx ds$ which has the extrema $1/(\lambda_{i+j} - t_i)$ if t_i is a fixed value with $\lambda_i < t_i < \lambda_{i+1}$. The integral form (3) is transformed into a form involving the kernel K only and related to an integral used by Temple. The approximate maximisation (minimisation) is carried out by successive approximation using the method of iterated kernels. The general boundary problem of the Sturm-Liouville equation, the first boundary problem of $y'' + \lambda(1+x)y = 0$ [note that y is missing in equation (60)] and the integral equation $y(x) = \lambda \int |x - s| y(s) ds$ are treated as examples. Here it is often possible to start the iteration with the known eigenfunction of an analytically soluble, related eigenvalue problem. *H. O. Hartley* (London).

Lehmann, N. Joachim. Beiträge zur numerischen Lösung linearer Eigenwertprobleme. II. Z. Angew. Math. Mech. 30, 1–16 (1950). (German. English, French and Russian summaries)

In the second paper the maximisation (minimisation) of (3) [see the preceding review] is considered in more detail, and $f(x)$ is replaced by a linear aggregate of specially chosen "coordinate functions" which form a complete system. This method simplifies the computational procedure and leads to improved gauges of the eigenvalues.

H. O. Hartley (London).

Seidel, W. P., Davison, B., and Kushneriuk, S. Influence of a small black cylinder upon the neutron density in an infinite non-capturing medium. National Research Council of Canada. Atomic Energy Project. Division of Research. **MT-207** (N.R.C. no. 1868), i+74 pp. (1948).

This paper discusses the solution of the integral equation considered in an earlier paper [National Research Council of Canada, Division of Atomic Energy, document no. **MT-135** (N.R.C. 1554) (1945); these Rev. **9**, 593]. The integral equation in question is denoted by (*) in the cited review; for the subsequent notation see that review. With a suitable normalization the asymptotic form of the required solution is $n(r) = 1 + B(a) \log (r/a) + O(\epsilon^{-r})$, where $B(a)$ is a constant depending on a only. This paper is devoted to the determination of $B(a)$ for $a \to 0$. It is shown that

$$(*)\quad a/B(a) = \tfrac{4}{3} + (1 - 16/(3\pi^2))a \log a - 0.2164a^2 + O(a^2 \log^2 a).$$

The method of solution consists in expressing the solution of the integral equation in the form (**) $n(r) = \sum_{m=0}^{\infty} n_m(r)$; $n_0(r) = 1$, and determining the $n_m(r)$'s successively as the solutions of the nonhomogeneous integral equations

$$n_m(r) = \frac{2}{\pi} \int_0^\infty r' n_m(r') dr' \int_{|r-r'|}^{r+r'} \frac{d\rho}{[\rho^2 - (r-r')^2]^{\frac{1}{2}}[(r^2+r')^2 - \rho^2]^{\frac{1}{2}}} \int_\rho^\infty K_0(t) dt + f_m(r) \qquad (m \geqq 1),$$

where $f_m(r)$ depends on $n_{m-1}(r)$. Interpreting the foregoing equation as governing the neutron density due to a cylindrically symmetrical system of sources $f_m(r)$ in a uniformly noncapturing medium, the authors show (using the known solution for an infinite line source) that the asymptotic form of $n_m(r)$ is given by

$$n_m(r) = -3\left(\int_0^\infty r f_m(r) dr \right) \log (r/a) + O(\epsilon^{-r}),$$

so that $B(a) = \sum_{m=1}^{\infty} B_m(a) = -3 \sum_{m=1}^{\infty} \int_0^\infty r f_m(r) dr$. With the explicit expression for $f_m(r)$ substituted in the expression for $B_m(a)$, it is shown that

$$B_m(a) = 3 \int_0^\infty r n_{m-1}(r) dr + \frac{6}{\pi} \int_a^\infty r dr \int_a^\infty r' n_{m-1}(r') dr' \times \int_{(r^2-a^2)^{\frac{1}{2}} + (r'^2-a^2)^{\frac{1}{2}}}^{r+r'} \frac{d\rho}{[\rho^2 - (r-r')^2]^{\frac{1}{2}}[(r+r')^2 - \rho^2]^{\frac{1}{2}}} \int_\rho^\infty K_0(t) dt.$$

For the case $m = 1$ and $n_0(r) = 1$, the integral defining B_1 can be simply reduced to give $B_1(a) = \frac{3}{4}a + O(a^2)$; this is the first term in the formula (*) quoted. The complete formula is obtained by considering, also, the equations for the case $m = 2$; the details of the calculation are, however, very involved.

The authors show that the procedure outlined is a convergent one in the sense that after n_1 the successive terms in (**) form a uniformly decreasing sequence of functions for a sufficiently small. *S. Chandrasekhar.*

Functional Analysis, Ergodic Theory

*von Neumann, John. **Functional Operators. II. The Geometry of Orthogonal Spaces.** Annals of Mathematics Studies, no. 22. Princeton University Press, Princeton, N. J., 1950. iii+107 pp. $2.25.

[For volume I, 1950, cf. these Rev. 11, 240.] This volume contains chapter XII (Linear spaces), chapter XIII (Linear operators), and chapter XIV (Commutativity, reducibility). As the subtitle of the volume indicates, the principal subject is not the algebraic and analytic theory of operators, but

rather their relation to the geometry of Hilbert space. Chapter XII treats the relations among various possible requirements on an inner product space (dimensionality, separability, completeness) and gives the standard measure-theoretic examples. Chapter XIII treats the natural elementary properties of and constructions based on operators (closure, extension, adjoint) in terms of their graphs. The main concern of chapter XIV is to provide the tools for a careful discussion of the matrices associated with operators.

P. R. Halmos (Chicago, Ill.).

Štraus, A. V. On generalized resolvents of a symmetric operator. Doklady Akad. Nauk SSSR (N.S.) **71**, 241–244 (1950). (Russian)

This paper is related to a series of papers by Neumark [cf. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 53–104, 277–318 (1940); **7**, 285–296 (1943); these Rev. **2**, 104, 105; **6**, 71] and Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 323–326 (1944); **52**, 651–654 (1946); these Rev. **6**, 131; **8**, 277] on spectral functions of symmetric operators on a Hilbert space $\mathfrak{H}$. Using the results from two of his previous papers [Doklady Akad. Nauk SSSR (N.S.) **67**, 611–614 (1949); **70**, 577–580 (1950); these Rev. **11**, 186, 442] the author obtains an expression for the generalized resolvents of a closed Hermitian operator A, whose domain is dense in $\mathfrak{H}$, for the case of arbitrary non-zero indices of deficiency, thus generalizing the results of Krein. In the notation of his previous papers [cf. these Rev. **11**, 186] the formula for the generalized resolvent R_λ is

$$R_\lambda=(A-\lambda I)^{-1}(I-P(\lambda))-\frac{1}{\lambda-\bar\lambda}P(\lambda)$$
$$+\frac{1}{\lambda-\bar\lambda}P(\lambda)[(\bar\lambda_0-\lambda)I-(\lambda_0-\lambda)F(\lambda)]$$
$$\times\{P(\lambda)[(\lambda_0-\lambda)I-(\lambda_0-\lambda)F(\lambda)]\}^{-1}P(\lambda),$$

where $\Im\lambda>0$, $\Im\lambda_0>0$, and $F(\lambda)$ is an arbitrary regular operator-function from $\mathfrak{M}(\bar\lambda_0)$ to $\mathfrak{M}(\lambda_0)$, with $\|F(\lambda)\|<1$. The author states further that to distinct $F(\lambda)$ correspond distinct R_λ, and that resolvents corresponding to self-adjoint extensions of the second kind in Neumark's sense [cf. the 1st cited reference] are characterized by the condition $\|F(\lambda)\varphi\|<\|\varphi\|$ for all $\varphi\in\mathfrak{M}(\bar\lambda_0)$, $\varphi\neq0$. *B. Crabtree.*

Halmos, Paul R. Commutativity and spectral properties of normal operators. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 153–156 (1950).

The author proves for the case of a bounded normal operator A in Hilbert space the fact that if the bounded operator B commutes with A, then B commutes with the adjoint A^* of A. The proof given here differs from the one given by B. Fuglede [Proc. Nat. Acad. Sci. U. S. A. **36**, 35–40 (1950); these Rev. **11**, 371] whose method yields the result for arbitrary normal A and a bounded B. (The author remarks that a minor modification of his method will cover the general case.) The key to the proof is the following theorem which seems to be new. For a bounded normal operator A let $\mathfrak{F}(A)=\{x\mid\|A^*x\|\leqq\|x\|\}$. For every complex number λ and every $\epsilon>0$, let $\mathfrak{F}(\lambda,\epsilon)=\mathfrak{F}((A-\lambda)/\epsilon)$. For every set M of complex numbers let $\mathfrak{F}(M,\epsilon)$ be the subspace spanned by all $\mathfrak{F}(\lambda,\epsilon)$ with λ in M. Let $\mathfrak{F}(M)=\bigcap_{\epsilon>0}\mathfrak{F}(M,\epsilon)$, and let $E(\lambda)$ be the resolution of the identity for A: $A=\int\lambda dE(\lambda)$. Then if $\mathfrak{E}(M)=\{x\mid(\int_M dE(\lambda))(x)=x\}$, it follows that, for every compact set M, $\mathfrak{E}(M)=\mathfrak{F}(M)$. This result is based on the "regularity" of the spectral measure $E(M)=\int_M dE(\lambda)$. *B. Gelbaum* (Minneapolis, Minn.).

Dunford, Nelson. Resolutions of the identity for commutative B^*-algebras of operators. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 51–56 (1950).

The author generalizes the simultaneous spectral resolution of a commutative family of normal operators as follows. Let $\mathfrak{A}$ be a commutative self-adjoint uniformly closed algebra of bounded operators on a (not necessarily separable) Hilbert space H. Denote by Λ the (compact Hausdorff) space of maximal ideals of $\mathfrak{A}$. It is proved that there exists for every Borel subset ϵ of Λ a projection E_ϵ on H such that the E_ϵ have the formal properties of a resolution of the identity and such that every $T\in\mathfrak{A}$ is of the form $\int_\Lambda f(\lambda)dE_\lambda$ where $f(\lambda)$ is the continuous function on Λ corresponding to T. The proof uses a general form of the F. Riesz representation of the regular Borel measures on Λ [cf. Kakutani, Ann. of Math. (2) **42**, 994–1024 (1941); these Rev. **3**, 205] and the Gelfand and Neumark representation of $\mathfrak{A}$ as the algebra of all continuous functions on Λ [Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 197–213 (1943); these Rev. **5**, 147].

F. I. Mautner (State College, Pa.).

Zwahlen, Robert. Opérateurs hermitiens à valeurs propres liées par une formule de récurrence rationnelle. C. R. Acad. Sci. Paris **230**, 352–353 (1950).

The author considers a Hermitian operator F in Hilbert space whose characteristic values λ_n satisfy a recurrence relation of classical polynomial type. Under this hypothesis there exists a recurrence operator S connecting the characteristic vectors y_n: $y_{n+1}=Sy_n$. The author states that F and S satisfy a related polynomial equation. Moreover, any solution S of the equation such that S_{y_0} is again a characteristic vector of F is a recurrence operator for F. No proofs are given. *W. F. Eberlein* (Madison, Wis.).

Fan, Ky. On a theorem of Weyl concerning eigenvalues of linear transformations. I. Proc. Nat. Acad. Sci. U. S. A. **35**, 652–655 (1949).

The author proves the following results on linear operators in an n-dimensional unitary space. (i) If the eigenvalues λ_i of a Hermitian operator H are arranged so that $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n$, and $1\leqq q\leqq n$, the sums $\lambda_1+\cdots+\lambda_q$ and $\lambda_{n-q+1}+\cdots+\lambda_n$ are respectively the maximum and minimum of $(Hx_1,x_1)+\cdots+(Hx_q,x_q)$ for all sets of q orthonormal vectors $x_1,\cdots,x_q$. (ii) If the eigenvalues λ_i of a normal operator N are arranged so that $|\lambda_1|\geqq|\lambda_2|\geqq\cdots\geqq|\lambda_n|$, and s,q are positive integers with $1\leqq q\leqq n$, then $|\lambda_1|^s+\cdots+|\lambda_q|^s$ is the maximum of

$$\|(UN)^sx_1\|^2+\cdots+\|(UN)^sx_q\|^2,$$

where U runs over all unitary operators and $x_1,\cdots,x_q$ over all sets of q orthonormal vectors. (iii) If A is any linear operator, s is a positive integer, the eigenvalues $\kappa_i^{(s)}$ ($\kappa_i^{(1)}=\kappa_i$) of $(A^s)^*A^s$ are arranged so that $\kappa_1^{(s)}\geqq\kappa_2^{(s)}\geqq\cdots\geqq\kappa_n^{(s)}$, and $1\leqq q\leqq n$, then $\kappa_1^{(s)}+\cdots+\kappa_q^{(s)}\leqq\kappa_1^s+\cdots+\kappa_q^s$. Extensions to completely continuous operators in Hilbert space are indicated. *F. Smithies* (Cambridge, England).

Kračkovskiĭ, S. N., and Gol'dman, M. A. On the principal part of a completely continuous operator. Doklady Akad. Nauk SSSR (N.S.) **70**, 945–948 (1950). (Russian)

Let $\mathfrak{A}$ be a completely continuous operator in Hilbert space H, $\mathfrak{A}(H)\subseteq H$. An eigenspace corresponding to an

eigenvalue λ of $\mathfrak{A}$ is finite-dimensional and is spanned by a canonical system of vectors $x_1, \cdots, x_n$ with $\lambda\mathfrak{A}x_1=x_1$, $\lambda\mathfrak{A}x_2=x_2+x_1, \cdots, \lambda\mathfrak{A}x_n=x_n+x_{n-1}$. The author represents $\mathfrak{A}$ as the sum of two completely continuous operators $\mathfrak{A}=\mathfrak{A}_1+\mathfrak{A}_2$, where $\mathfrak{A}_1$ has the same eigenvalues and the same canonical systems as $\mathfrak{A}$ has, and $\mathfrak{A}_2$ has no eigenvalues; $\mathfrak{A}_1(H)\subseteq L$ where L is spanned by all eigenspaces of $\mathfrak{A}$, $\|\mathfrak{A}_1\|\leqq\|\mathfrak{A}\|$, $\|\mathfrak{A}_2\|\leqq\|\mathfrak{A}\|$, $\mathfrak{A}_2\mathfrak{A}_1=0$. The operator $\mathfrak{A}_1$ is termed the principal part of $\mathfrak{A}$. The resolvent of the principal part of $\mathfrak{A}$ is the principal part of the resolvent of $\mathfrak{A}$. Some other properties of the principal part are proved.

O. M. Nikodým (Gambier, Ohio).

Grinblyum, M. M. Operational integral in a Banach space. Doklady Akad. Nauk SSSR (N.S.) **71**, 5–8 (1950). (Russian)

This continues two earlier notes [same Doklady **70**, 749–752, 941–944 (1950); these Rev. **11**, 525]. The author deals with a special function of intervals of the real line, finitely additive and of bounded variation, whose values are commuting projections of a reflexive Banach space E into itself. (These are the spectral functions of the second of the preceding notes.) Integrals are defined in the usual way [see Gowurin, Fund. Math. **27**, 254–268 (1936) for a general definition] for the collection K of uniform limits of real-valued step functions. It is noted that this integral for a fixed interval function $P(\ )$ is a continuous homomorphism of the ring K into the ring of all bounded linear operators from E to E. An improper integral is defined, for example, for functions of the class K on every finite interval; it is asserted that, for each such f, the improper integral $\int_{-\infty}^{\infty} f dP$ is a closed operator with dense domain.

M. M. Day.

Mysovskih, I. P. On the convergence of L. V. Kantorovič's method of solution of functional equations and its applications. Doklady Akad. Nauk SSSR (N.S.) **70**, 565–568 (1950). (Russian)

The author supplements results of Kantorovič on the functional equation (1) $P(x)=0$ [same Doklady (N.S.) **59**, 1237–1240 (1948); Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 89–185 (1948); these Rev. **9**, 537; **10**, 380], where the operator $P(x)$ has the properties assumed by Kantorovič of transforming X into Y (both Banach spaces) and of being twice differentiable in the sense of Fréchet. The results, patterned after those of Kantorovič, concern the existence and the uniqueness of a solution of (1), using one or the other of the sequences of approximation (2) $x_{k+1}=x_k-[P'(x_k)]^{-1}P(x_k)$; (7) $x_{k+1}=x_k-[P'(x_0)]^{-1}P(x_k)$. [The numbering of equations is that used by the author.] Four theorems are stated (one of them proved) of a general nature, and two theorems on applications.

A typical general theorem (compare with the first Kantorovič theorem [these Rev. **9**, 537]) is as follows. Suppose (i) $\|P(x_0)\|\leqq\eta$ for the initial approximation x_0; (ii) the inverse operator $\Gamma(x)=[P'(x)]^{-1}$ exists at each point of the sphere (3) $\|x-x_0\|\leqq H(h)B_0\eta$ (where $H(h)=\sum_{k=0}^{\infty}(h/2)^{2^k-1}$, $0\leqq h<2$), and throughout (3), $\|\Gamma(x)\|\leqq B$, with $\|\Gamma(x_0)\|\leqq B_0\leqq B$; (iii) (4) $\|P''(x)\|\leqq K$ in (3); (iv) $h=BB_0K\eta<2$. Then in the sphere (3) equation (1) has a solution x^* to which the approximating sequence (2) converges with a rapidity given by $\|x_k-x^*\|\leqq H(h)B_0\eta(h/2)^{2^k-1}$. Moreover, if in (iv) the number 2 is replaced by α ($\cong 1.13$), a root of $hH(h)=2$, then x^* is unique in the sphere $\|x-x_0\|<2h^{-1}B_0\eta$ if (4) holds in this sphere.

One application of the general theorems is to a system of n algebraic equations in n variables. The other deals with the integral equation (12) $x(s)=\int_0^1 K(s,t)f(t,x(t))dt$. Taking $X=Y$ as the space of continuous functions on $[0,1]$, the following result is given. Suppose (i) the kernel $K(s,t)f_x'(t,x_0(t))$, where the initial approximation $x_0(t)$ is continuous, has the resolvent $G(s,t)$ with $\max\int_0^1|G(s,t)|dt\leqq B$; (ii) $\max|x_0(s)-\int_0^1 K(s,t)f(t,x_0(t))dt|\leqq\eta$; (iii) $K(s,t)$ is measurable on $[0,1;0,1]$, continuous in s and satisfies the inequality $K^2(s,t)\leqq K_1(t)$, where $K_1(t)$ is summable on $[0,1]$; (iv) in the region G of points (t,x) for which $0\leqq t\leqq 1$, $|x-x_0(t)|\leqq 2(B+1)\eta$, the function $f(t,x)$ is measurable, and $f_x'^2$ exists and its square does not exceed $D(t)$ in G, where $D(t)$ is summable on $[0,1]$; (v) $f(t,x_0(t))$ and $f_x'(t,x_0(t))$ are of class L_2; (vi) $(B+1)^2K\eta\leqq\frac{1}{4}$, where $K=\{\int_0^1 K_1(t)dt\cdot\int_0^1 D(t)dt\}^{\frac{1}{2}}$. Then equation (12) has a unique solution $x^*(s)$ in the sphere $\|x-x_0\|\leqq 2(B+1)\eta$, and to it the sequence of approximations (2) converges.

I. M. Sheffer (State College, Pa.).

Schäfke, Friedrich Wilhelm. Über einige unendliche lineare Gleichungssysteme. Math. Nachr. **3**, 40–58 (1949).

The author considers the system of linear equations (*) $\sum\alpha_{ik}\xi_k=\eta_i$ ($i=1,2,\cdots$). Let B be a Banach space and $\bar{B}$ the conjugate space. If $\{h_k\}$, $\{\bar{h}_k\}$ ($h_k\varepsilon B$, $\bar{h}_k\varepsilon\bar{B}$) is a biorthogonal set ($\bar{h}_i(h_k)=\delta_{ik}$), if the series $\sum\alpha_{ik}h_k=b_i$ converge in B, and if one considers only those solutions of (*) for which $\sum\xi_k\bar{h}_k=\bar{x}$ converges in $\bar{B}$, then (*) can be written as $\bar{x}(b_i)=\eta_i$ ($i=1,2,\cdots$). In order to solve a system (*) in this way one has to answer questions like the following about a given sequence $\{f_k\}$ in a Banach space: Is $\{f_k\}$ minimal? Complete? A base? For which sequences of numbers $\{\alpha_k\}$ is $\sum\alpha_kf_k$ convergent? In this paper B is a Hilbert space H [compare the following review]. The above questions lead the author to the introduction of bounded bases and bounded biorthogonal sets in H. Let (**) $\{f_k\}$, $\{g_k\}$ be a biorthogonal set (that is, $(f_i,g_k)=\delta_{ik}$). If both $\{f_k\}$ and $\{g_k\}$ form a (Schauder) base of H, and if $\sum\alpha_kf_k$ and $\sum\alpha_kg_k$ each converge if and only if $\sum|\alpha_k|^2$ converges, then (**) is called a bounded biorthogonal system. A base $\{f_k\}$ which together with its (unique) adjoint sequence $\{g_k\}$ ($(f_i,g_k)=\delta_{ik}$) forms a bounded biorthogonal set is called a bounded base.

(1) The following result extends Paley and Wiener's theorem [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, p. 100; compare the remarks made by R. P. Boas, Trans. Amer. Math. Soc. **48**, 467–487 (1940); these Rev. **2**, 80]. If $\{h_k\}$ is a bounded base, and if $\{f_k\}$ satisfies (***) $\|\sum\alpha_k(h_k-f_k)\|\leqq\lambda\|\sum\alpha_kh_k\|$ ($\lambda<1$, fixed) for all finite sets of α_k, then $\{f_k\}$ is a bounded base, and conversely. (2) The main result of the paper is: if $\{h_k\}$ is a complete orthonormal set, then the sequences $\{f_k\}$, $\{g_k\}$ form a bounded biorthogonal set if and only if the matrix $\|(f_i,g_k)\|$ is bounded and has $\|(h_i,g_k)\|$ as its (left and right) bounded inverse. (A matrix $\|\alpha_{ik}\|$ is called bounded if $\sum|\xi_k|^2<\infty$, $\eta_i=\sum\alpha_{ik}\xi_k$, imply $\sum|\eta_i|^2<\infty$.) (3) It follows from (2) that if $\{h_k\}$ is a complete orthonormal set and if $\{f_k\}$ satisfies $\sum|(a,f_k-h_k)|^2\leqq\lambda\|a\|^2$ for all $a\varepsilon H$ (λ fixed <1), then $\{f_k\}$ is a bounded base. This extends S. H. Hilding's result according to which $\{f_k\}$ is complete if $\sum\|f_k-h_k\|^2<1$ [Ark. Mat. Astr. Fys. **35A**, no. 38 (1948); these Rev. **11**, 174]. The above results, mainly (2), are used to investigate some bases in L_2, to study certain infinite matrices and to discuss a system of equations connected with a physical problem. The latter system is too complicated to be reproduced here; some of the other results are: (4) if δ is not an integer, then $\{\cos(2k+\delta)\xi\}$ ($k=0,\pm1,\cdots$) is a bounded base in

$L_2(0, \pi)$; (5) the matrix $M(\delta) = \|(-1)^{k-i} \sin \pi\delta/\pi(k-i+\delta)\|$ $(i, k = \cdots, -1, 0, 1, \cdots)$ is bounded and satisfies

$$M(\delta_1)M(\delta_2) = M(\delta_1+\delta_2).$$

J. Korevaar (Lafayette, Ind.).

Schäfke, Friedrich Wilhelm. Das Kriterium von Paley und Wiener im Banachschen Raum. Math. Nachr. **3**, 59–61 (1949).

The author proves the following extension of Paley and Wiener's theorem [compare the first paragraph and (1) of the preceding review]. Let $\{h_k\}$ be a base of B, and let $\{f_k\} \subset B$ satisfy (***) ($\lambda < 1$, fixed) for all finite sets of α_k. Then $\{f_k\}$ is also a base of B. [It might be mentioned that this result is due to Boas; see (1) of the preceding review.] Hence both $\{h_k\}$ and $\{f_k\}$ have unique adjoint sequences $\{\bar{h}_k\}$, $\{\bar{f}_k\}$ $(\bar{h}_i(h_k) = \bar{f}_i(f_k) = \delta_{ik})$. If $\{\bar{h}_k\}$ is a base in B, then so is $\{\bar{f}_k\}$. Finally, $\sum \alpha_k h_k$ and $\sum \alpha_k f_k$ converge at the same time; also $\sum \beta_k \bar{h}_k$ and $\sum \beta_k \bar{f}_k$. It is not known whether the converse (that is, the conclusions for $\{h_k\}$, $\{\bar{h}_k\}$ from the data on $\{f_k\}$, $\{\bar{f}_k\}$, while (***) holds) is true except in the trivial case $\lambda < \frac{1}{2}$. *J. Korevaar* (Lafayette, Ind.).

Kozlov, V. Ya. On bases in the space $L_2[0, 1]$. Mat. Sbornik N.S. **26(68)**, 85–102 (1950). (Russian)

All functions mentioned in this review belong to $L_2(0, 1)$. Let $\{g_n\}$, $\{h_n\}$ be two biorthogonal systems with $\|h_n\| = 1$. Orthogonalisation of $\{g_n\}$ yields $\{t_n\}$. Let $S_n(f) = \sum_{k=1}^{n} (f, g_k)h_k$ and $B_n = \sup \|S_n(f)\|/\|f\|$. Theorem: $\{h_n\}$ is a basis of $L_2(0, 1)$ if and only if there is a complete set $\{g_n\}$ biorthogonal to $\{h_n\}$ such that $B_n < K$ $(n = 1, 2, \cdots)$. Other necessary and sufficient conditions for $\{h_n\}$ to be a basis are deduced. They are fairly complicated sets of inequalities involving the $S_n(f)$ of certain classes of functions f. An example is given of a nonorthogonal basis such that $B_n = S_n(r_n)$. For such a basis $\{h_n\}$ there are two constants K, L such that $K\|f\|^2 < \sum_{k=1}^{n} (f, g_k)^2 < L\|f\|^2$ and $\{h_n - B_n r_n\}$ is orthogonal.

W. H. J. Fuchs (Ithaca, N. Y.).

Šilov, G. E. Description of a class of normed rings of functions. Mat. Sbornik N.S. **26(68)**, 291–310 (1950). (Russian)

A ring R of continuous functions on an interval $a \leqq t \leqq b$ is said to possess one generator provided R is equal to the completion of the ring of polynomials $P(t)$ with respect to the norm $\|P(t)\|$ given in R. The ring R is said to be of type C provided its norm is equivalent to the norm $\|x(t)\|' = \sup_{a \leqq s \leqq b} \|x(t)\|_s$, where $\|x(t)\|_s = \inf \|y(t)\|$, the infimum being taken over all $y(t)$ in R each of which coincides with $x(t)$ in some neighborhood of the point s. The ring C of all continuous functions $x(t)$ on $a \leqq t \leqq b$ under the norm $\|x(t)\| = \max_{a \leqq t \leqq b} |x(t)|$ is a ring of type C with one generator as is also the ring D_1 of all continuous functions $x(t)$ on $a \leqq t \leqq b$ which possess a continuous derivative $x'(t)$ under the norm $\|x(t)\| = \max_{a \leqq t \leqq b} \{|x(t)| + |x'(t)|\}$. In the present paper a characterization is obtained for all rings of type C with one generator which lie between D_1 and C. It turns out that such rings are obtained by completing the ring of polynomials $P(t)$ under a norm

$$\|P(t)\| = \sup_{a \leqq t \leqq b} \{|P(t)| + \alpha(t)|P'(t)|\},$$

where $\alpha(t)$ is a function, called the characteristic function of the ring, which is nonnegative, upper semi-continuous and satisfies a certain admissibility condition too complicated to be stated here. Conversely, every such characteristic function gives rise to a ring R_α of the desired kind. An intrinsic characterization of functions in R_α is obtained in terms of a notion of differentiation with respect to $\alpha(t)$ defined as follows. Denote by F_n the set of all t such that $\alpha(t) \geqq 1/n$. Then a function $f_\alpha'(t)$ is said to be a "generalized derivative of $f(t)$ with respect to $\alpha(t)$" provided $f_\alpha'(t)$ is continuous on each of the sets F_n, $f(t)$ is absolutely continuous on each interval Δ where $1/\alpha(t)$ is summable and $f_\alpha'(t) = f'(t)$ almost everywhere on Δ. This generalized derivative has the same elementary properties as the ordinary derivative and coincides with it when $\alpha(t) \equiv 1$. A necessary and sufficient condition for a continuous function $f(t)$ to belong to R_α is that the generalized derivative $f_\alpha'(t)$ exist and $\max_{t \in F_n} |f_\alpha'(t)| = o(n)$.

C. E. Rickart (New Haven, Conn.).

Hamburger, Hans Ludwig. Remarks on the Dirac δ-operator. Proc. Cambridge Philos. Soc. **45**, 489–494 (1949).

Let the linear space F of real or complex functions on the interval Δ contain the sequence $\{f_n(s)\}$. The author establishes the nonexistence of a function $I(s, t)$ on $\Delta \times \Delta$ such that $\int_\Delta I(s, t)f(t)dt = f(s)$ almost everywhere for every f of F in the following cases: (a) $\Delta = (0, 1)$; $f_n(s) = s^n$; (b) $\Delta = (0, \infty)$; $(f_n) = (\exp(-s/2), \exp(-s), s\exp(-s), s^2\exp(-s), \cdots)$; (c) $\Delta = (-\infty, \infty)$;

$$(f_n) = \left(\exp(-s^2/2), \exp(-s^2/2)\int_{-\infty}^{s} \exp(-u^2/2)du, \exp(-s^2), s\exp(-s^2), \cdots\right).$$

The procedure involves a preliminary integration by parts and comparison with the Stieltjes integral expression of the appropriate δ-operator. *W. F. Eberlein* (Madison, Wis.).

Morse, Marston. Bilinear functionals over $C \times C$. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 41–48 (1950).

This article is a brief and yet very clear report on the basic motivation and central results of a program undertaken jointly by the author and W. Transue. The typical feature of the program is to replace the Vitali variation of a function $k(x, y)$, used in previous literature in the theory of bilinear functionals, double Fourier series, and other problems, by a less restrictive concept of variation due to Fréchet. This results in improvements over previous results obtained by various authors. *T. Radó.*

Maler, A. G. On central trajectories and a problem of Birkhoff. Mat. Sbornik N.S. **26(68)**, 266–290 (1950). (Russian)

For a dynamical system D let O denote the operation of taking the closure of the union of the limit trajectories and let $b(D)$ denote the ordinal number of the central trajectories defined as the least number of times the operation O must be performed to obtain invariance under O. Then $b(D)$ is necessarily an ordinal number of the first or second class. The principal result of this paper is the following. If a is an ordinal number of the second class, then it is possible to construct a dynamical system D in the 3-dimensional solid torus such that D satisfies a Lipschitz condition and such that $b(D) > a$. Except for analyticity, this verifies a conjecture of G. D. Birkhoff. *W. H. Gottschalk.*

Barbašin, E. A. On a condition for the existence of a secant surface. Doklady Akad. Nauk SSSR (N.S.) **70**, 365–368 (1950). (Russian)

In a manifold M_n consider a flow, given by $dx_i/dt=X_i$, where the X_i are the components of a vector field over M_n. A secant surface or cross section is a surface F such that, after multiplying the vector field by a suitable positive function, every point passes periodically through F, with a fixed period. A differential $\omega=\sum P_i dx_i$ is admissible if $\sum P_i X_i>0$ everywhere. According to a remark of G. D. Birkhoff, a cross section clearly exists if there exists on M_n an angular coordinate φ (i.e., a multiple-valued function, whose values differ by multiples of 2π), with an admissible differential $d\varphi$. Theorem: if M_n possesses an admissible closed differential, then there exists a cross-section. For the proof one shows that the (classical) existence of a Poisson-stable trajectory implies that the given differential is not homologous to zero, and then, by using de Rham's theorems, one constructs a differential, all of whose periods (with respect to a base for the 1-cycles) are rational and positive; the integral of this differential is then an angular coordinate with admissible differential. A second theorem is stated: the existence of a (not necessarily smooth) cross section implies the existence of an admissible differential. It follows, via the proof of the first theorem, that there exists an angular coordinate (of class C^2); this replaces an argument of Birkhoff's [Dynamical Systems, Amer. Math. Soc. Colloquium Publ., v. 9, New York, 1927, p. 144] which is apparently incomplete. *H. Samelson* (Ann Arbor, Mich.).

Calculus of Variations

Faedo, Sandro. Il calcolo delle variazioni per gli integrali su intervalli infiniti. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 94–125 (1949).

This is a continuation of previous work of the author [see Pont. Acad. Sci. Comment. **8**, 319–421 (1944); these Rev. **10**, 130]. The paper includes two theorems on unique determination of extremals, a set of sufficient conditions for the first variation to be finite, and, for the case of a quadratic integrand, a set of sufficient conditions for the vanishing of the first variation. *L. M. Graves* (Chicago, Ill.).

Giuliano, Landolino. Sulla continuità degli integrali curvilinei del calcolo delle variazioni. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 76–80 (1949).

For parts I and II cf. the same journal (8) **4**, 39–45, 183–189 (1948); these Rev. **10**, 259. This note is a summary of another paper [Ann. Scuola Norm. Super. Pisa (3) **1** (1947), 161–187 (1949); these Rev. **11**, 253].

L. M. Graves (Chicago, Ill.).

Dedecker, Paul. Sur un problème inverse du calcul des variations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 63–70 (1950).

Using properties of Cartan's exterior differential forms defined in certain spaces $E_{(s)}$ of elements of contact, the author establishes a necessary and sufficient condition for a system of n ordinary differential equations of order s in the dependent functions $y^1, \cdots, y^n$ to be identical with the Euler-Lagrange equations of a problem of the calculus of variations. *W. T. Reid* (Evanston, Ill.).

Dufresnoy, Jacques. Remarques sur les extrémales du problème fondamental de calcul des variations. Bull. Tech. Univ. Istanbul **1**, 1–10 (1948). (French. Turkish summary)

For nonparametric simple integral problems of the calculus of variations

$$\text{(I)} \qquad \int \phi(y_1, \cdots, y_n, y_1', \cdots, y_n')dx,$$

with integrand function ϕ independent of x, the author presents a related treatment of the reduction of such problems to equivalent ones by transformations of the form $y_i=F_i(X)$, $i=1, \cdots, n$, $dx=\Lambda(F_1, \cdots, F_n, F_1', \cdots, F_n')dX$, and the extremum properties with respect to certain associated integrals that are possessed by the projections of extremals of (I) on the $(y_1, \cdots, y_n)$-space. In particular, dynamical systems of Liouville type belong to the class of problems reducible to quadratures by transformations of the kind considered; an illustration of the second class of problems treated is afforded by the relationship between Hamilton's principle and Jacobi's form of the principle of least action. *W. T. Reid* (Evanston, Ill.).

Leighton, Walter. Principal quadratic functionals. Trans. Amer. Math. Soc. **68**, 253–274 (1949).

This paper is a continuation of a study undertaken earlier by M. Morse and the author [same Trans. **40**, 252–286 (1936)]. The functional with which this paper has mainly to do is $J[y(x)]=\int_0^b[r'(x)y'^2(x)-p(x)y^2(x)]dx$ where $r(x)$, $p(x)$ are continuous functions in $0<x\leqq b$. In general J is supposed singular at $x=0$. The function $p(x)$ is always assumed to be of one sign near $x=0$. For the problem of both end points fixed, let us call A the class of all functions $y(x)$, $0\leqq x\leqq b$, continuous on $0\leqq x\leqq b$, $y(0)=y(b)=0$, which (α) in each interval (a, b), $0<a<b$ are absolutely continuous with derivatives $y'(x)$ which are L^2-integrable. If $J[a, b; y]=\int_a^b(ry'^2-py^2)dy$ let $i=\liminf J(a, b; y)$ when $a\to+0$ for all $y(x)\in A$. For the problem of only one end point fixed, let us call B the class of all functions $y(x)$, $0<x\leqq b$, continuous on $0<x\leqq b$, $y(b)=0$, satisfying (α) and let $j=\liminf J(a, b; y)$ when $a\to+0$ for all $y(x)\in B$. Necessary and also sufficient conditions are stated in order that $i\geqq 0$ or $j\geqq 0$. They involve the same singularity condition introduced in the previous paper, the Euler differential equation of J and a new notion of focal point of the singular y-axis defined as the limit of the focal point $\xi=\xi(x_0)$ of the straight line $x=x_0$ when $x_0\to 0$. Extensions are given to the case of both end points singular and the functional $\int_0^b(ay'^2+2byy'+cy^2)dx$. *L. Cesari* (Lafayette, Ind.).

Sigalov, A. G. The existence of an absolute minimum for double integrals of the calculus of variations in parametric form. Doklady Akad. Nauk SSSR (N.S.) **70**, 769–772 (1950). (Russian)

The theorem named in the title has in the past shown itself to be highly intractable. In these notes the author indicates the main steps of its demonstration. Let K be the unit square $0\leqq u^1, u^2\leqq 1$; let L^2 be the class of surfaces having representations $x=f(u)$ $(x=(x^1, x^2, x^3); u\in K)$ which are absolutely continuous in the sense of Tonelli and have finite Dirichlet integrals. The integrand $F(x, A)$ is assumed defined and continuous for all $x\in\mathfrak{C}^3$ and all $A=(A^1, A^2, A^3)$, and is assumed positively homogeneous and convex in A for each fixed x. If $T\in L^2$ is represented by $f(u)$, with properties just named, the three Jacobians are denoted by A^1, A^2, A^3;

for each open or closed subset G of K, the integral $\int\int_G F(f(u), A)du$ is denoted by (f, G, F), or by (f, G) when $F(x, A)$ happens to be $\|A\|$; the integral to be discussed is $J(T)=(f, K, F)$. Given a Jordan curve Γ, a surface T is "admissible" if it belongs to L^2 and is bounded by Γ. The number μ_1 is defined to be inf $J(T)$ for all admissible T; $\mu_0(L^2)$ is the infimum, for all sequences $\{T_n\}$ of L^2 surfaces with boundaries Γ_n tending to Γ, of lim inf $J(T_n)$; $\mu_0(\mathfrak{P})$ is analogous, the surfaces T_n being required to belong to the class $\mathfrak{P}$ of polyhedra. The existence theorem announced is as follows. Assume Γ and F as above; assume further

$$M \equiv \sup_{\|A\|=1,\ x\in K^2} F(x, A) < \infty, \quad m \equiv \inf_{\|A\|=1,\ x\in K^2} F(x, A) > 0.$$

Then if admissible surfaces exist, there is an admissible surface T such that $J(T)=\mu_1=\mu_0(\mathfrak{P})=\mu_0(L^2)$.

The lower semicontinuity of $J(T)$ is assumed known [cf. E. J. McShane, Ann. of Math. (2) **33**, 460–484 (1932)]. It remains to exhibit a convergent minimizing sequence. For a region G, define the "simply connected cover" $\bar{G}$ to be the least set containing the interiors of all simple closed polygons lying in G [this is the substitute definition stated in the note reviewed below]. Let $\bar{G}'$ be the boundary of $\bar{G}$. Let $l\{f, C\}$ denote the length of the image under f of the curve C in K. For each region $G\subset K$ define $Q\{f, G\}=(f, \bar{G})/l\{f, \bar{G}'\}^2$, if the denominator is not zero. Define $\lambda^i_-\{f, G\}$ to be the difference of the infima of f^i on G and on its boundary G'; $\lambda^i_+\{f, G\}$ the corresponding difference of suprema;

$$\lambda^i\{f, G\}=\max[\lambda^i_-\{f, G\}, \lambda^i_+\{f, G\}];$$
$$\lambda\{f, G\}=\max[\lambda^i\{f, G\}, i=1, 2, 3].$$

Given a representation $x=f(u)$, $u\in K$, of a surface T, a region $G\subset K$ is principal if for some real c and some one i of the numbers 1, 2, 3 the region G is a component of $E[f^i<c]$ or of $E[f^i>c]$ and its boundary G' does not meet the boundary K' of K. A region G is simple if it is bounded by a simple closed polygon such that $G'\cap K'$ is empty or else connected and not consisting of one point. The author states the following criterion for compactness. In order that a sequence $\{P_n\}$ of polyhedra have a subsequence converging in the sense of Fréchet to a surface in L^2, it is sufficient that the following conditions be satisfied. (1) The areas of the P_n are bounded. (2) The P_n have boundary curves without multiple points, and a subsequence of these boundary curves tends in the sense of Fréchet to a simple closed curve. (3) To each $\epsilon>0$ corresponds Q^* such that: each P_n has a piecewise linear representation $x=f_n(u)$, $u\in K$, for which (*) $\lambda\{f_n, G\}<\epsilon$ holds whenever G is a principal region and either $l\{f_n, G'\}=0$ or $Q\{f_n, G\}>Q^*$. (4) To each $\epsilon>0$ corresponds $\eta>0$ such that each P_n has a piecewise linear representation $x=f_n(u)$, $u\in K$, for which (**) $(f_n, G)<\epsilon$ whenever G is a simple region and $l\{f, G'-G'\cap K'\}<\eta$ and Osc $\{f, G'\cap K'\}<\eta$.

From this the representations f_n seem to depend on ϵ and to be possibly different in (3) and (4). Apparently this is not actually intended by the author. Also, the criterion is not perfectly suited to its use in the paper reviewed below. The reviewer found it possible to establish the following criterion. Conditions (1) and (2) are as above. To each P_n corresponds a piecewise linear representation $x=f_n(u)$, $u\in K$ with properties: (3′) to each $\epsilon>0$ corresponds n_ϵ' and Q^* such that (*) holds for $n>n_\epsilon'$; (4′) to each $\epsilon>0$ corresponds $\eta>0$ and n_ϵ'' such that (**) holds for $n>n_\epsilon''$. A similar amendment seems essential in the statement of the properties of the auxiliary functions g_n constructed in the proof. These are piecewise conformally represented; equicontinuity on the boundary K' follows from a theorem of the reviewer [Trans. Amer. Math. Soc. **35**, 716–733 (1933)] and equicontinuity on K by use of the (ϵ, δ)-gratings of L. C. Young [ibid. **64**, 317–335 (1948), in particular, lemma 3.1; these Rev. **10**, 109]. *E. J. McShane.*

Sigalov, A. G. On the existence of an absolute minimum for double integrals of the calculus of variations. Doklady Akad. Nauk SSSR (N.S.) **71**, 617–620 (1950). (Russian)

Assume $F(x, A)$ defined and continuous for all x in a neighborhood of the closure of a convex open set D_1 and for all A, and to be positively homogeneous of degree 1 and convex in A for each fixed x. In the definition of admissibility in the preceding review the author now adds the requirement that T lie in $\bar{D}_1$, and in the definitions of $\mu_0(L^2)$ and $\mu_0(P)$ he adds the requirement that every neighborhood of $\bar{D}_1$ contain almost all T_n. The conclusion of the existence theorem is asserted to remain valid. Two other theorems are stated asserting conditions which assure the existence of minimizing surfaces of small diameter. The contents of this note consist almost entirely of indications of the possibility of constructing a minimizing sequence satisfying the criterion of compactness stated in the preceding note. For T of finite Lebesgue area $J(T)$ is defined to be the infimum of lim inf $J(T_n)$ for all sequences of surfaces of L^2 tending to T. This extension is shown to coincide with Cesari's integral $J_C(T)$. *E. J. McShane* (Charlottesville, Va.).

Theory of Probability

Dumas, Maurice. Sur une relation entre les valeurs typiques et les écarts typiques d'ordres divers d'une loi de probabilité. C. R. Acad. Sci. Paris **230**, 813–815 (1950).

The author studies typical values of discrete distributions. He shows that a law having finite typical écart of order $\alpha>1$ has the property that the difference between its median and its typical value of order α is at most equal to the typical écart of order α. *E. Lukacs.*

Salvi, Filippo. Estensione di alcuni teoremi classici del calcolo delle probabilità. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 282–308 (1949).

In this posthumous paper classical theorems (such as Poisson's theorem) referring to simple alternatives are extended to the multinomial case. *E. Lukacs.*

Friede, Georg. Über Reziprozitätsbeziehungen in der Wahrscheinlichkeitsrechnung. Z. Angew. Math. Mech. **30**, 65–72 (1950). (German. English, French and Russian summaries)

By "reciprocity relations" the author seems to mean those having a dual interpretation, as in the expression of the incomplete binomial summation (the probability of at least x occurrences in n Bernoulli trials) by the incomplete gamma function ratio and the Bayes interpretation of the latter as the probability that the unknown probability of the event lies between 0 and p, given that $x-1$ events have appeared in n trials. By elaborate calculations from an urn scheme, that is, with drawings with replacements from an urn containing R kinds of balls and ball i with frequency p_i, the author shows first that the incomplete binomial summation

is also a sum of Pascal probabilities:

$$\sum_{s=x}^{n}\binom{n}{s}p^sq^{n-s}=\sum_{s=x}^{n}\binom{s-1}{x-1}p^xq^{s-x},\qquad p+q=1$$

[a relation apparently going back to Montmort; cf. Czuber, Wahrscheinlichkeitsrechnung, v. 1, 3d ed., Teubner, Leipzig-Berlin, 1914, pp. 126–127], and finally that the integrand of the incomplete gamma function ratio, which he writes: $h(p, x, n)=n\binom{n-1}{x-1}p^{x-1}q^{n-x}dp$ has an interpretation, independent of Bayes' considerations, as the probability that the unknown probability of the event lies in $(p, p+dp)$ when at least x events appear in n trials. The development of the statistical consequences of the last remark in another paper is promised. *J. Riordan* (New York, N. Y.).

Kanellos, S. G. On the independence of two events and the regularization of a random variable. Bull. Soc. Math. Grèce **24**, 85–102 (1949). (Greek. English summary)

Let A and B be two independent events with probabilities p and q. In a sequence of n independent trials let X, Y, and Z be the numbers of occurrences, respectively, of A, B, and AB. From a consideration of the trinominal distribution the author finds that for $n\to\infty$ the probability tends to .999 that Z lies within the interval $pY+qX-pqn\pm3.2952(npq(1-p)(1-q))^{\frac{1}{2}}$. Various simple statistical tests are derived. *W. Feller* (Princeton, N. J.).

Consael, R. Sur le schéma de Pólya-Eggenberger à deux variables aléatoires. Assoc. Actuair. Belges. Bull. no. **55**, 11–23 (1949).

Consider an urn containing Np red, Nq black, and Nr white balls. Drawings are made and after each drawing $1+\delta$ balls of the same color as the one drawn are placed in the urn. This generalizes the Pólya-Eggenberger scheme in which there are only balls of two colors. Let x_n, y_n, z_n be the number of balls of the three colors drawn in n drawings. The limiting bivariate distributions are obtained for n large under the following circumstances: (1) p and q constant, $\delta=0$; (2) np and nq constant, $\delta=0$; (3) p, q and $n\delta/N$ constant; (4) np and nq constant; (5) p, nq and $n\delta/N$ constant; (6) p, q and δ/n constant; (7) $p\to0$, $q\to0$, $(\delta/N)\to0$, $np^2\to\infty$, $nq^2\to\infty$; pN/δ and qN/δ positive constants.

J. L. Doob (Urbana, Ill.).

Risser, R. Note relative aux tirages contagieux. Assoc. Actuair. Belges. Bull. no. **55**, 25–51 (1949).

The author considers the same probability scheme as that in the preceding review. He finds an approximate expression for the frequency function of x_n, y_n, z_n good for a wide range of values of the various parameters of the problem, and then finds several limiting distributions for large n [see also the preceding review]. *J. L. Doob* (Urbana, Ill.).

Borel, Émile. Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaîne. C. R. Acad. Sci. Paris **230**, 591–593 (1950).

R. Coustal [same vol., 431–432 (1950); these Rev. **11**, 402] published the first 1033 digits of $2^{\frac{1}{2}}$. The author discusses certain features which may lead one to suspect their randomness. No actual tests are applied. *W. Feller.*

Mack, C. The expected number of aggregates in a random distribution of n points. Proc. Cambridge Philos. Soc. **46**, 285–292 (1950).

The author considers n points randomly distributed in a certain bounded domain of the plane. A k-aggregate is formed by k points which can be covered by a unit square or a triangle of a given shape. The author evaluates the expected number of k-aggregates neglecting border-effect. Similar formulas are then derived for a higher number of dimensions. *W. Feller* (Princeton, N. J.).

Rényi, Alfréd. Some remarks on independent random variables. Hungarica Acta Math. **1**, no. 4, 17–20(1949).

Let $X_1, X_2, \cdots, X_n$ be independent bounded random variables all having the same distribution function $F(x)$. Assume furthermore that

$$\int_{-\infty}^{\infty}xdF(x)=0,\quad \int_{-\infty}^{\infty}x^2dF(x)=\sigma^2.$$

Let M_n be the greatest lower bound of those M for which $\Pr\{|X_1+\cdots+X_n|<M\}=1$. The principal result of this paper is the inequality $M_n\geqq\frac{1}{2}n\sigma^2/\Phi(0)$, where

$$\Phi(0)=\int_{-\infty}^{0}F(x)dx.$$

The result is best possible as can be seen by considering Bernoulli variables. *M. Kac* (Ithaca, N. Y.).

Gnedenko, B. V. On the domain of attraction of the normal law. Doklady Akad. Nauk SSSR (N.S.) **71**, 425–428 (1950). (Russian)

The known necessary and sufficient condition for a distribution function $F(x)$ to belong to the domain of attraction of the normal distribution is shown to be equivalent to the condition that for $t\to0$ one has $\psi(st)/\psi(t)\to s^2$ for every fixed s where $\psi(t)=\log|f(t)|$ and $f(t)$ is the characteristic function of $F(x)$. For sums of independent random variables the analogous condition requires that $\sum\psi_k(st)/\sum\psi_k(t)\to s^2$. Let the distribution $F(x)$ be a step function with jumps at $\pm nh$ (where h is the largest number with this property), and suppose that $F(x)$ belongs to the domain of attraction of the normal distribution. Then $F^{n*}(x)$ is again a step function, and it is shown that the probability mass at nh is asymptotically given by the variation of the normal distribution over $(n+\frac{1}{2})h$. *W. Feller* (Princeton, N. J.).

Hinčin, A. Ya. On sums of positive random variables. Doklady Akad. Nauk SSSR (N.S.) **71**, 1037–1039 (1950). (Russian)

Let $\Omega(x)$ be the density function of the sum of n nonnegative, independent chance variables, each with the same density function $\omega(x)$, and expected value a. Let $0<a'<a$, $n\to\infty$, $x=o(n^{\frac{1}{2}})$. Then $e^{-\beta x}\Omega(na'+x)/\Omega(na')\to1$ where β is the root of $-\varphi'(\beta)/\varphi(\beta)=a'$ and $\varphi(\beta)=\int_0^\infty\omega(x)e^{-\beta x}dx$. Applications of this result to several physical questions are mentioned. *J. Wolfowitz* (New York, N. Y.).

***Smirnov, N. V. Limit distributions for the terms of a variational series.** Trudy Mat. Inst. Steklov. **25**, 60 pp. (1949). (Russian)

Let $Y_{(k,n)}$ be the kth in order of size of n independent chance variables with the same distribution function $F(x)$. This paper deals at great length with the asymptotic distribution $\Phi(x)$ of $(Y_{(k,n)}-b_n)/a_n$, where $a_n>0$ and b_n are suitable constants. Most of the results fall into two classes: (a) $n\to\infty$, k constant, (b) $n\to\infty$, $k/n\to\lambda$ (a constant), $n^{\frac{1}{2}}(k/n-\lambda)\to0$. It is impossible to describe the results within the bounds of a brief review. A sample is given by theorem 7:

for the case (b) $\Phi(x)$ can only be one of the following types:

$$(1)\qquad \Phi_\alpha^1(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{cx^\alpha} e^{-z^2/2}dz, \quad x\geqq 0,\ c>0;$$
$$=0, \quad x<0;$$

$$(2)\qquad \Phi_\alpha^2(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{-c|x|^\alpha} e^{-z^2/2}dz, \quad x<0,\ c>0;$$
$$=1, \quad x>0;$$

$$(3)\qquad \Phi_\alpha^3(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{-c_1|x|^\alpha} e^{-z^2/2}dz, \quad x<0,\ c_1>0;$$
$$=\tfrac{1}{2}+(2\pi)^{-\frac{1}{2}}\int_{0}^{c_2x^\alpha} e^{-z^2/2}dz, \quad x>0,\ c_2>0;$$

$$(4)\qquad \Phi^4(x)=0, \quad x\leqq -1;$$
$$=\tfrac{1}{2}, \quad -1<x\leqq 1;$$
$$=1, \quad x>1.$$

J. Wolfowitz (New York, N. Y.).

***Fréchet, Maurice. Statistical Self-Renewing Aggregates. A Course of Lectures Delivered at the Faculty of Science, Fouad I University. Edited with five appendices and a bibliography by J. Boulos Simaika.** Fouad I University Press, Cairo, 1949. v+126+6 pp. (English. Arabic summary)

Part I, "The empirical study of renewal," introduces the fundamental notions. Part II, "The renewal theory," discusses the renewal coefficients for the case of a discrete process assuming that the mortality function has a polynomial or rational generating function. Lotka's method is also discussed. Many examples are given. Two appendices reproduce Simaika's papers [Skand. Aktuarietidskr. **30**, 121–129 (1947); Proc. Math. Phys. Soc. Egypt **3** (1947), no. 3, 15–23 (1948); these Rev. **9**, 452; **10**, 542].

W. Feller (Princeton, N. J.).

Siraždinov, S. H. The ergodic principle for nonstationary Markov chains. Doklady Akad. Nauk SSSR (N.S.) **71**, 829–830 (1950). (Russian)

Let $M_1, M_2, \cdots$ be the successive matrices of one-step transition probabilities of a Markov chain with n states. Then 1 is a characteristic value of M_k. Let $c_k\leqq 1$ be the maximum of the moduli of the remaining characteristic values (excluding the value 1 only if it is simple). The author shows that if $n=2$ the condition (1) $\sum(1-c_k)=\infty$ is necessary and sufficient for

$$(2)\qquad \lim_{\nu\to\infty}[M_k\cdots M_m]_{ij}-[M_k\cdots M_m]_{i'j}=0$$

(all i, i', j, k). On the other hand the author gives an example of Sarymsakov which shows that (1) is not sufficient for (2) if $n>2$. It is shown that if d_k is the determinant of M_k, the condition $\sum(1-|d_k|)<\infty$ is (rather trivially) necessary for (2). Condition (2) is usually described as the condition of ergodicity, and Kolmogorov had conjectured that (1) was necessary and sufficient for (2).

J. L. Doob (Urbana, Ill.).

Chung, Kai-Lai. An estimate concerning the Kolmogoroff limit distribution. Trans. Amer. Math. Soc. **67**, 36–50 (1949).

Let $X_1, X_2, \cdots$ be a sequence of independent random variables having a common distribution function $F(x)$ which is assumed to be continuous. Let $nF_n(x)$ denote the number of the variables $X_1, \cdots, X_n$ whose values do not exceed x, and write

$$d_n=\sup_{-\infty<x<\infty}|nF_n(x)-F(x)|, \quad \Phi(\lambda)=\sum_{-\infty}^{\infty}(-1)^\nu e^{-2\nu^2\lambda^2},$$

$P(d_n\leqq\lambda n^{\frac{1}{2}})=\Phi(\lambda)+R_n(\lambda)$. The well-known result due to Kolmogoroff that $R_n(\lambda)\to 0$ as $n\to\infty$ for any fixed $\lambda>0$ is here improved by considering the order of $R_n(\lambda)$ for large n, and also allowing λ to depend on n. Thus it is proved, e.g., that, if $A_0>0$ is an arbitrary fixed constant, and $(A_0\log n)^{-1}<\lambda(n)<A_0\log n$, we have

$$R(\lambda(n))=O((\log n)^{5/2}n^{-1/10}).$$

From this result it is deduced that, for an increasing $\lambda(n)$ tending to infinity, the probability $P(d_n>\lambda(n)n^{\frac{1}{2}}$ infinitely often) is 0 or 1 according as $\sum n^{-1}\lambda^2(n)e^{-2\lambda^2(n)}$ converges or diverges.

H. Cramér (Stockholm).

Linnik, Yu. V., and Sapogov, N. A. Multiple integrals and local laws for inhomogeneous Markov chains. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 533–566 (1949). (Russian)

In the first part of the paper Sapogov proves the central limit theorem for a sequence of h-dimensional vectors whose values depend on the states of a discrete Markov chain. The number of states is supposed to be finite, but may vary with time. Various uniformity conditions are imposed on the transition probabilities so that S. Bernstein's classical results can be used. [The essential part of the results was announced in Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 190–192 (1949); these Rev. **11**, 189. Cf. also Doklady Akad. Nauk SSSR (N.S.) **58**, 193–196, 1905–1908 (1947); these Rev. **9**, 293, 361.] In the second part Linnik considers a Markov chain with $h+1$ states $\epsilon_0, \epsilon_1, \cdots, \epsilon_h$, and stationary transition probabilities. For $1\leqq\nu\leqq h$ let $N_\nu(n)$ be the number of passages during the first n trials through ϵ_ν. It is supposed that the chain belongs to the type A of Doeblin's classification which means essentially that the variance of each $N_\nu(n)$ tends to infinity together with n. It is shown that the joint distribution of $N_\nu(n)$ is asymptotically normal. The proof is exceedingly complicated and depends on intricate estimates.

W. Feller (Princeton, N. J.).

Sapogov, N. A. On the law of large numbers for dependent random variables. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 145–154 (1950). (Russian)

Consider a triangular array of random variables X_{jn}, $j\leqq n$, $n=1, 2, \cdots$. The author states conditions under which the weak law of large numbers applies. One such condition implies the uniform boundedness of conditional expectations and that the covariance of X_{jn} and X_{kn} tends to zero as $|j-k|\to\infty$, uniformly in n. The case of finite Markov chains is given special attention.

W. Feller.

Moran, P. A. P. The spectral theory of discrete stochastic processes. Biometrika **36**, 63–70 (1949).

Let $\{x_n, -\infty<n<\infty\}$ be a stationary stochastic process, where x_n is a p-dimensional vector variable. The author calculates the effect on the spectral density of the x_n process of taking moving averages, and shows that the Slutzky sinusoidal limit theorem and its extension by Romanovsky [Rend. Circ. Mat. Palermo **57**, 130–136 (1933)] are thereby easily derived.

J. L. Doob (Urbana, Ill.).

Hanner, Olof. Deterministic and non-deterministic stationary random processes. Ark. Mat. **1**, 161–177 (1950).

Let $\{x_t, -\infty<t<\infty\}$ be a family of random variables (stochastic process) which is stationary in terms of its first and second moments. Wold [A Study in the Analysis of Stationary Time Series, Uppsala, 1938] proved, for t integral valued, a decomposition theorem which has applications in

the prediction theory of this type of process, that x_t can be expressed as the sum of two orthogonal components, one of which is completely deterministic, and the other completely nondeterministic. The author proves the analogous theorem in the continuous parameter case. The theorem is proved by geometric Hilbert space methods, without the use of the spectral theory of stationary stochastic processes.

J. L. Doob (Urbana, Ill.).

Karhunen, Kari. Über die Struktur stationärer zufälliger Funktionen. Ark. Mat. 1, 141–160 (1950).

The author studies the same problem as that in the preceding review, taking up the problem from a more analytic point of view, and using spectral theory. He finds the necessary and sufficient conditions for the various types of processes from the point of view of prediction theory (deterministic and so on) in terms of the spectral densities of the processes. These conditions are known [Krein, C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 306–309 (1945); Wiener, Extrapolation, Interpolation, and Smoothing of Stationary Time Series, with Engineering Applications, Wiley, New York, 1949; these Rev. 7, 61; 11, 118].

J. L. Doob (Urbana, Ill.).

Mathematical Statistics

***Cochran, William G., and Cox, Gertrude M. Experimental Designs.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. ix+454 pp. $5.75.

While this is not a mathematics book at all, it is worth a brief comment here because it will be of considerable interest to most mathematical statisticians. The book consists of a thorough exposition of the practical aspects of experimental designs. The first three chapters give a preliminary discussion of the role of statistics in experimentation, of the general nature of experimental designs, and of the numerical analysis of experimental data. Then follows a detailed presentation of all the important designs: randomized blocks, latin and greco-latin squares, factorial designs, balanced and partially balanced incomplete blocks, lattices and lattice squares, together with various combinations of these designs using different methods of confounding. The special properties of the designs are clearly described and the analyses of the designs are completely illustrated by experiments that have actually been performed and with all the data and computations included.

The book is then an excellent reference work, and also a much better textbook for a nonmathematical course than Fisher's "The Design of Experiments," although it does not have any problems for the student to solve. The reviewer had hoped to find a chapter on how, in general, one goes about designing experiments; this is more of an art than a science, at least for the present, but certainly two of the foremost experts in this field could have had some very useful things to say in such a chapter. This is not to imply that authors refrain from giving advice, for there is much sage counsel to be found throughout the book.

A. M. Mood (Santa Monica, Calif.).

Hammersley, J. M. Further results for the counterfeit coin problems. Proc. Cambridge Philos. Soc. 46, 226–230 (1950).

Given n coins of which one is counterfeit and all are of unknown weight, a chemical balance, and a set of standard weights, one is to find the counterfeit coin and find the weights of the true coins as well as the counterfeit coin. The least number of weighing operations required is shown to be the least integer m satisfying $m \geqq \log_3(2n+1)$.

A. M. Mood (Santa Monica, Calif.).

Khargonkar, S. A. The estimation of missing plot value in split-plot and strip trials. J. Indian Soc. Agric. Statistics 1, 147–161 (1948).

The author derives explicitly all the formulae needed for the analysis of variance of the designs in the title, when one value is missing, and also indicates the proper procedure for more than one missing value. The estimates of the missing values are obtained by an extension of Yates's method [Empire J. Experimental Agriculture 1, 129–142 (1933)].

H. B. Mann (Berkeley, Calif.).

Narain, R. D. A new approach to sampling distributions of the multivariate normal theory. II. J. Indian Soc. Agric. Statistics 1, 137–146 (1948).

The author extends the method of part I [same vol., 59–69 (1948); these Rev. 10, 387] for computing distributions of multivariate analysis. This method involves the use of the easily obtainable joint distribution of the partial regression coefficients of x_r on x_{r-s} given $x_1, \cdots, x_{r-s-1}$ and of the unexplained sum of squares of the regression of x_r on $x_1, \cdots, x_{r-1}$. With slight variations the author gives another derivation of the multiple correlation coefficient. This method has the advantage of giving as a by-product the conditional distribution of the correlation coefficient in the case where the dependent variables are given and in the case where the independent variable is given. These conditional distributions have the same form as has been previously observed by others. These distributions are applied to obtain the distribution of the D^2 statistic. The power function of two general hypotheses involving the use of the T-test are shown to be different.

H. Chernoff.

Skellam, J. G. The distribution of the moment statistics of samples drawn without replacement from a finite population. J. Roy. Statist. Soc. Ser. B. 11, 291–296 (1949).

The author is interested in the moments of statistics in samples of size n drawn without replacement from a finite population of size N. Let the sample values be x_j, $j=1, 2, \cdots, n$; let $\sum_{j=1}^{n} x_j^a = S_a$, the sample value, and $\sum_{j=1}^{N} x_j^a = M_a$, the corresponding population value; then formulas are derived by a simplified procedure for $E\{\omega\}$, where $\omega = S_a, S_aS_b, S_a^2, \cdots$ to all those of weight 4 and of weight 5; $S_a^4S_b$, $S_a^2S_b^2$, $S_a^2S_bS_c$, $S_a^2S_b^2S_c$. These check the results of Sukhatme [Sankhyā 6, 363–382 (1944); these Rev. 6, 8]. [The following references should be added to those listed by the author: Dwyer, Ann. Math. Statistics 8, 21–65 (1937); 9, 1–47, 97–132 (1938).]

L. A. Aroian.

Krishna Iyer, P. V. The first and second moments of some probability distributions arising from points on a lattice and their application. Biometrika 36, 135–141 (1949).

A lattice is a rectangular array of points in any Euclidean space. To each point is assigned at random one of a finite number of colors. Two ways of assigning colors are considered: (a) "free sampling," where the probabilities are identical and the assignment is independent for each point, (b) "nonfree sampling," which refers to a conditional distribution under (a), the condition being that the total number of points of each color is fixed. The present paper gives the first and second moments of the number of joins

between adjacent points which are (a) of the same color, (b) of two specified colors, (c) of different colors, for free and nonfree sampling, and for two- and n-dimensional lattices. [Results for two- and three-dimensional lattices in the case of free sampling have been given by the author in Nature **162**, 333 (1948)]. Applications to statistical problems are discussed. *J. Wolfowitz* (New York, N. Y.).

Patnaik, P. B. The non-central χ^2- and F-distributions and their applications. Biometrika **36**, 202–232 (1949).

Numerous approximations are presented for the probability integral of the noncentral χ'^2 based on n observations with k linear restraints,

$$\chi'^2=\sum_{i=1}^{n}\xi_i^2-\sum_{j=1}^{k}\rho_j^2,$$

where ξ_i are normally and independently distributed with means a_i and unit variance. The linear restraints are $\sum_{i=1}^{n}c_{ji}\xi_i=\rho_j$ $(j=1, \cdots, k)$, where $\sum_{i=1}^{n}c_{ji}c_{ki}=\delta_{jk}$ (Kronecker's delta). The parameter of the distribution of χ'^2 is $\lambda=\sum a_i^2-\sum_j(\sum a_ic_{ji})^2$. These approximations are as follows: (i) $(n+\lambda)\chi'^2/(n+2\lambda)\sim\chi^2$ with $(n+\lambda)^2/(n+2\lambda)$ degrees of freedom. (ii) $\{2\chi'^2(n+\lambda)/(n+2\lambda)\}^{\frac{1}{2}}\sim$normal with mean $\{[2(n+\lambda)^2/(n+2\lambda)]-1\}^{\frac{1}{2}}$ and unit variance. (iii) Closer approximations using higher moments of (i) and (ii). Comparisons of the results using these approximations are made with a few exact values computed by Fisher and Garwood.

These results for χ'^2 are applied to the following problems. (i) Power of the χ^2-test (with no linear restraints) computed for $n=2(1)10(2)20$, $\lambda=\sum a_i^2=2(2)20$, and type I error $\alpha=0.05$. (ii) Power of goodness-of-fit tests for both simple and composite hypotheses. If we have a noncentral $\chi_1'^2$ with parameter λ and n_1 degrees of freedom and a central χ_2^2 with n_2 degrees of freedom, $F'=n_2\chi_1'^2/n_1\chi_2^2$ is called a noncentral F. Then F'/k is approximated by a central F with $n=(n_1+\lambda)^2/(n_1+2\lambda)$ and n_2 degrees of freedom, where $k=(n_1+\lambda)/n_1$. Comparisons are made of approximate values of the probability integral with some exact values computed by Tang.

Approximate power curves are computed for some analysis of variance tests with a one-way classification, using both random and systematic models. It is shown that the power curve for the random model is below that for the systematic model. Contours of equal power (β) for the systematic model are presented as follows: (i) $\beta=0.5$, $\alpha=0.05$, $n_1=1(1)10$, $n_2=8$ to ∞, and $\lambda=0$ to 22. (ii) $\beta=0.9$, $\alpha=0.05$, $n_1=1(1)10$, $n_2=8$ to ∞, and $\lambda=0$ to 45. *R. L. Anderson.*

Bateman, G. I. The characteristic function of a weighted sum of non-central squares of normal variates subject to s linear restraints. Biometrika **36**, 460–462 (1949).

This note presents the characteristic function of a weighted noncentral χ'^2 subject to s linear restraints, of which the χ'^2 considered by Patnaik [see the preceding review] is a special case. The weighted χ'^2 is $\chi'^2=\sum_{j=1}^{n}c_jx_j^2$, where the x_j are normally and independently distributed with means a_j and unit variances. *R. L. Anderson* (Raleigh, N. C.).

Nair, K. R. A further note on the mean deviation from the median. Biometrika **36**, 234–235 (1949).

This is a follow-up of a previous note [Biometrika **34**, 360–362 (1947); these Rev. **9**, 363]. For estimating σ of a normal population, the coefficient of variation for the mean deviation from the median is almost the same as for the mean deviation from the mean, when the sample size n satisfies $2\leqq n\leqq 10$. *R. L. Anderson* (Raleigh, N. C.).

Huzurbazar, V. S. Probability distributions and orthogonal parameters. Proc. Cambridge Philos. Soc. **46**, 281–284 (1950).

This paper is concerned with the problem of choosing such parameters that the information matrix is diagonal. The motivation, attributed here to Jeffreys, is to render the maximum likelihood equations tractable to a iterative method. It is pointed out that the problem amounts to that of covering a Riemannian manifold with n mutually orthogonal families of hypersurfaces. Translation and scale parameters are investigated and also, as special cases of a moderately general method, Pearson families of types II, VII, and III, the latter being treated approximately.

L. J. Savage (Chicago, Ill.).

Roy, S. N., and Bose, P. K. On the construction of an unbiassed and most powerful critical region out of any given statistic. Calcutta Statist. Assoc. Bull. **1**, no. 4, 177–182 (1948).

Let T be a given statistic whose frequency function depends on a parameter θ. For testing $\theta=\theta_0$ against $\theta=\theta_1$ the most powerful among the tests which depend only on T is obtained by the usual Neyman-Pearson method. It is pointed out that the most powerful test is necessarily unbiased. A technique for determining the critical region is described. Examples are given. *W. Hoeffding* (Chapel Hill, N. C.).

Mosteller, Frederick, and Tukey, John W. Significance levels for a k-sample slippage test. Ann. Math. Statistics **21**, 120–123 (1950).

The authors study in relation to a previous article by Mosteller on a k-sample slippage test based on samples of equal size [same Ann. **19**, 58–65 (1948); these Rev. **9**, 454] the same problem in the case where the samples are of unequal size, and derive exact as well as approximate significance levels. *K. R. Buch* (Copenhagen).

Divatia, V. V. A note on sequential exponential tests. J. Indian Soc. Agric. Statistics **2**, 86–93 (1949).

This is a straightforward application of Wald's theory of sequential analysis [Ann. Math. Statistics **16**, 117–186 (1945); these Rev. **7**, 131] to derive tests of hypotheses concerning the scale parameter of an exponential distribution. Various typographical errors should be noted.

D. G. Chapman (Seattle, Wash.).

Fay, Edward A. Grouping of observations for fitting regression curves. NAVORD Rep. **1057 (NOTS 177)**, U. S. Naval Ordnance Test Station, Inyokern, Calif., vi+30 pp. (1948).

Assuming that the regression of y on x is a polynomial of given degree and that the variance of y is constant for all x in the range $(0, T)$, the author considers the problem: If one may choose r sure values of x equally spaced on the range at each of which to make n observations of y so that the total number of observations is $N=nr$, how can one make the choice: (a) to minimize the maximum variance of residuals at any point in $(0, T)$; (b) to minimize the average variance of residuals over $(0, T)$? The author gives detailed results for polynomials through degree 4. As he notes case (a) was handled with greater generality by K. Smith [Biometrika **12**, 1–85 (1918)]. The principal tool is an application of results in the paper of F. N. David and J. Neyman on the Markoff theorem on least squares [Statist. Res. Mem. London. **2**, 105–116 (1938)]. *C. C. Craig.*

TOPOLOGY

Boyer, Jean M., and Hall, D. W. A note on Peano spaces. Proc. Amer. Math. Soc. 1, 231–232 (1950).

Let I denote a straight line interval. By addition of a certain countable set of straight line intervals to I, each meeting I in a single point and meeting one another in at most a point of I, a dendrite L is produced having the following property: if N and P are Peano continua such that $N-P=0\neq P-N$ and $g(I)=N$ any mapping, then there exist a homeomorphism $h(I)=I$ and a mapping $f(L)=P$ having, among others, the following properties: (i) on I, $f=gh$; (ii) $f^{-1}(P-N)\subset L-I$. *R. L. Wilder.*

Fort, M. K., Jr. Essential and non essential fixed points. Amer. J. Math. 72, 315–322 (1950).

Let X be a compact metric space with distance function d, and with the fixed point property. Let X^X be the set of all continuous mappings of X into itself, metrized by $\rho(f, g)=\sup d(f(x), g(x))$, $x\in X$. A point $x_0\in X$ fixed under f is defined by the author to be an essential fixed point of f if corresponding to each neighborhood $U(x_0)$ there is an $\epsilon>0$ such that every g satisfying $\rho(f, g)<\epsilon$ has a fixed point in $U(x_0)$. It is shown that if $f\in X^X$, arbitrarily near f there is a $g\in X^X$ all of whose fixed points are essential. Another main result is the following. Let X be a compact manifold with the fixed point property, let x_0 be an isolated fixed point of f. Suppose N is a closed cell-neighborhood of x_0 in which x_0 is the only fixed point of f, and let h be a homeomorphism of N onto the solid unit sphere T_n in E_n such that $h(x_0)$ is the origin. For $P\in E_n$, let $\phi(P)=P/\|P\|$. Denote by f_R^* the mapping of the surface S_{n-1} of T_n into itself defined for a small R by $f_R^*(u)=\phi(Ru-hfh^{-1}(Ru))$ for $u\in S_{n-1}$. Then x_0 is an essential fixed point of f if and only if f_R^* is an essential mapping. *S. B. Myers* (Ann Arbor, Mich.).

Myškis, A. D. The definition of a boundary by means of continuous mappings. Mat. Sbornik N.S. 26(68), 225–227 (1950). (Russian)

Myškis, A. D. On the equivalence of certain methods of definition of a boundary. Mat. Sbornik N.S. 26(68), 228–236 (1950). (Russian)

The notation will be as in an earlier paper [Mat. Sbornik N.S. 25(67), 387–414 (1949); these Rev. 11, 382, cited as I]. The notions introduced there are now specialized by the author in various ways. (a) Let $y=\phi(x)$ map continuously a topological space R onto a dense subset $\phi(R)$ of a Hausdorff space Π and let $\Pi_0=\Pi-\phi(R)$. Let $\mathfrak{M}$ be the set of classes M_y of open sets U in R such that there exists a neighbourhood V of y for which $\phi^{-1}(V)\subset U$, and let $\mathfrak{M}$ be topologized as in (I). Then $\mathfrak{M}$ is a boundary of R in the wide sense. Sufficient conditions are given for $\mathfrak{M}$ to be so in the narrow sense. If Π is regular $\mathfrak{M}$ is homeomorphic to Π_0.

(b) Given in a space R (i) a contracting sequence $G_1, G_2, \cdots$ of open sets, or (ii) a sequence $\{x_n\}$ of points, we term "associated class of open sets" the class of the open sets G in R such that $G\supset G_n$ for some n, or that $x_n\in G$ for all large n. If R is a simply-connected plane domain, the author extends the notion of a Carathéodory chain of subdomains $\{G_n\}$ to mean a contracting sequence of simply-connected subdomains whose boundaries relative to R are connected and have positive distances from one another but zero distances from the boundary of R; he shows that this extension is immaterial to the Carathéodory notion of end, here defined as the class of open sets associated with a Carathéodory chain, and to the notion of prime end, here defined as an end not contained (as proper subclass) in any other end. If R is a domain in a finite-dimensional Euclidean space, a Perkins chain $\{G_n\}$ means a contracting sequence of open sets in R with diameters tending to zero, whose absolute boundaries meet that of R while their boundaries relative to R coincide with the boundaries relative to R of their closures and have positive distances from one another; a Mazurkiewicz sequence $\{x_n\}$ means a sequence of points of R such that $\rho(x_n, x_m)\to 0$ as $n, m\to\infty$, where $\rho(p, q)$ denotes the minimum diameter of continua in R joining the points p, q; the author proves that classes of open sets associated with Perkins chains are identical with those associated with Mazurkiewicz sequences; the set of these classes topologized as in (I), the Perkins boundary of R, is thus equivalent to the corresponding Mazurkiewicz definition. Finally when R is a general topological space, the author compares his own definition with a descriptive boundary, also due to Mazurkiewicz [Fund. Math. 33, 177–228 (1945); these Rev. 8, 47]. *L. C. Young* (Madison, Wis.).

Finzi, Arrigo. Sulla generazione di una trasformazione finita assegnata su una curva chiusa mediante una trasformazione infinitesima. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 688–694 (1949).

Let $g(x)$ define a homeomorphism T of a circle C onto itself. Assume that the rotation number k of T is irrational. The author announces that T is generated by an infinitesimal transformation if d^2g/dx^2 exists and satisfies a Lipschitz condition and if the coefficients in the continued fraction development of k satisfy a certain condition as to growth. He also considers the simultaneous generation of a family $T(\theta)$ of homeomorphisms by a family of infinitesimal transformations. Proofs are omitted. *P. A. Smith.*

Aleksandrov, P. S. On the dimension of closed sets. Uspehi Matem. Nauk (N.S.) 4, no. 6(34), 17–88 (1949). (Russian)

The author explains that the present paper is a reworking of the material of another memoir [Math. Ann. 106, 161–238 (1932)]. It was in that memoir that he introduced the concept of homology-dimension of a compactum and of dimension according to a "variable modulus." [See chapter VIII of Hurewicz and Wallman, Dimension Theory, Princeton University Press, 1941; these Rev. 3, 312; the Russian translation of this text is cited as a general reference in the paper.] The revision was undertaken in order to make the earlier work more readily understandable by basing it upon current "text-book" knowledge. The author lists the sections of his "Combinatorial Topology" [OGIZ, Moscow-Leningrad, 1947; these Rev. 10, 55] which are to be regarded as preliminary to the paper and these, of course, are rather numerous. The exposition avoids the use of cohomology theory. *L. Zippin* (Flushing, N. Y.).

Sitnikov, K. On the reduction of dimension by means of a mapping with a given set of fixed points. Uspehi Matem. Nauk (N.S.) 4, no. 6(34), 89–90 (1949). (Russian)

The author gives a proof of the following theorem, supplementing a result of Alexandrov in the paper reviewed above. Let E be an arbitrary r-dimensional set contained in a Euclidean or a Hilbert space R. Let E' be a closed subset of E of dimension not exceeding $r-2$. There exists a con-

tinuous map f of E into R, in which the image $f(E)$ of the set E is of dimension less than r and in which all points of E' remain fixed. *L. Zippin* (Flushing, N. Y.).

Whitehead, J. H. C. Note on suspension. Quart. J. Math., Oxford Ser. (2) 1, 9–22 (1950).

The crude part of Freudenthal's suspension theorem, namely,

$$E\colon \pi_{r-1}(S^{n-1}) \approx \pi_r(S^n), \qquad r<2n-2,$$
$$E\pi_{2n-3}(S^{n-1}) = \pi_{2n-2}(S^n),$$

where E denotes the suspension operation, is generalized as follows. Let $Y=X\cup\{e_\lambda^n\}$ be a Hausdorff space which consists of an arcwise connected subspace X and a set of open cells $\{e_\lambda^n\}$. Each e_λ^n shall be open in Y and its closure shall be the image of an n-element E_λ^n under a map $\phi_\lambda\colon (E_\lambda^n, \dot{E}_\lambda^n)\to(Y, X)$ such that $\phi_\lambda | E_\lambda^n - \dot{E}_\lambda^n$ is a homeomorphism onto e_λ^n. Let a whisker $p_\lambda q_\lambda$ be attached to a point $q_\lambda \varepsilon \dot{E}_\lambda^n$ for each λ and Q' be the disjoint union of these $p_\lambda q_\lambda \cup E_\lambda^n$. Let Q denote the space obtained by identifying all the points p_λ with a new point p_0. Let $P=U_\lambda(p_0q_\lambda\cup \dot{E}_\lambda^n)$. Let $x_0 \varepsilon X$ and let $\phi\colon (Q, P, p_0)\to(Y, X, x_0)$ be a map such that $\phi|E_\lambda^n=\phi_\lambda$. Let $g_r\colon \pi_r(Q, P)\to\pi_r(Y, X)$ be the homomorphism induced by ϕ. The generalized theorem asserts that, if $\pi_s(X)=0$ for $s=1, \cdots, k$ $(1\leqq k<n)$, then $g_r\colon \pi_r(Q, P)\approx\pi_r(Y, X)$ $(r=2, \cdots, n+k-2)$, and g_{n+k-1} is onto. The proof is built upon two suspension lemmas and does not depend on Freudenthal's theorems. Hence it gives a correct and readable proof of the crude part of Freudenthal's results. As to the delicate part of the suspension theorems, the author reduces his generalized case to the one studied by Freudenthal. *S. T. Hu* (Princeton, N. J.).

Martinelli, Enzo. Un'osservazione sopra un teorema fondamentale della teoria degl'integrali in una varietà topologica. Boll. Un. Mat. Ital. (3) 4, 348–352 (1949).

A proof of de Rham's existence theorem is sketched, under the assumption that dual bases Γ^i_p, Δ^j_{n-p} for p- and $(n-p)$-cycles exist with the following property: $\Gamma^i_p \cap \Delta^j_{n-p}$ is empty if $i\neq j$, and a single point P^i if $i=j$; for every point $\bar{P}^i$ on Γ^i_p, sufficiently close to P^i, there exists a cycle $\bar{\Delta}^i_{n-p}$, homotopic to Δ^i_{n-p}, which meets Γ^i_p exactly in $\bar{P}^i$; the whole cycle $\bar{\Delta}^i_{n-p}$ is assumed to depend differentiably on $\bar{P}^i$, so that the collection of $\bar{\Delta}$'s forms a "tube" in the manifold. It is then possible (and sufficient for the theorem), by starting from a function which is 0 outside the tube and constant on the $\bar{\Delta}$'s, to construct a form ω^i_p, which is closed, and has a nonzero period with respect to Γ^i_p only. In elementary cases the cycles actually have the assumed behavior. *H. Samelson* (Ann Arbor, Mich.).

***Hirsch, Guy. La géométrie projective et la topologie des espaces fibrés.** Topologie algébrique, pp. 35–42. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

This paper contains a discussion of results previously announced by the author [C. R. Acad. Sci. Paris 223, 528–530 (1946); these Rev. 8, 166]. The methods of proof of the principal theorems are outlined very briefly.

W. S. Massey (Providence, R. I.).

Young, Gail S., Jr. On the factors and fiberings of manifolds. Proc. Amer. Math. Soc. 1, 215–223 (1950).

The main purpose of the paper is to investigate the problem raised by Montgomery and Samelson as to whether the Euclidean n-space E^n is a fibre bundle with a compact fibre. Partial results are obtained by means of some considerations concerning the 1- and 2-dimensional factors of manifolds. The main results can be summarized as follows. (1) There is no fibering of E^n, $n>1$, by a compact fibre of dimension 1 or $n-1$; (2) there is no fibering of E^n, $n>2$, by a compact fibre F of dimension $n-2$ onto a base space of dimension 2; and (3) there is no fibering of E^n by a fibre F of dimension 2. When the dimension n is small, these assertions lead to complete solution of the problem; namely, there is no fibering of E^n, $2\leqq n\leqq 4$, by a compact fibre of dimension greater than 0. For the case $n=5$, there is no fibering of E^5 by a compact fibre of positive dimension except possible dim $F=3$ and dim $B>2$.

Among other technical theorems, the following generalization of a theorem due to Borsuk plays an important rôle throughout the paper: If $A\times B$ is a manifold with dim $A=1$ or 2, then A is a manifold; A has a boundary if and only if $A\times B$ does. As a preliminary, an elementary theorem is proved which asserts that if a fibre bundle is connected and locally connected, then the components of the fibre are all homeomorphic, a theorem which apparently has escaped notice in the literature. *S. T. Hu.*

***Hopf, H. Sur les champs d'éléments de surface dans les variétés à 4 dimensions.** Topologie algébrique, pp. 55–59. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

This is a topological discussion without proofs of continuous fields of surface elements in an oriented closed differentiable V^4, which have but a finite number of singularities. (Such fields always exist.) Any such field F defines a 4-cocycle Γ with coefficients in $\pi_3(U)$, U being the space of oriented planes in 4-space. Since U can be expressed as $S^2\times S^2$, the coefficients may be taken as pairs of integers; hence Γ is characterized by such a pair $\Gamma\cdot V^4$. For $V^4=S^4$, $\Gamma\cdot V=(-1, 1)$. The pair is independent of F if and only if the second Betti number of V^4 is 0. Given F and F' (made to agree on 1-cells), there is a corresponding difference cocycle $\Delta(F, F')$. Then (with cup products)

$$\Gamma'=[\Delta(F', F)]^2+\Delta(F, \bar{F})\Delta(F', F)+\Gamma,$$

$\bar{F}$ being obtained from F by an antipodal transformation in each S^2. Since $\Delta(F', F)$ can be chosen arbitrarily, this shows the possible range of $\Gamma'\cdot V^4$. The proof depends on a lemma on the difference between the Hopf invariants of mappings of S^3 into S^2. Some instances of the theorem are given. If V^4 can be expressed as a complex manifold, then the first term of the pair is always 0. *H. Whitney.*

***Cartan, Henri. Sur la notion de carapace en topologie algébrique.** Topologie algébrique, pp. 1–2. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

Some remarks on the author's cohomology theory for locally compact spaces, as yet unpublished. Comparison is made with the Eilenberg-Steenrod theory, and the origin of his ideas is given. *H. Whitney* (Cambridge, Mass.).

Lihtenbaum, L. M. On the topology of surfaces. Uspehi Matem. Nauk (N.S.) 5, no. 1(35), 214–216 (1950). (Russian)

Let R be a two-dimensional polyhedron whose faces are simple polygons, with the property denoted by (α) that

each edge of any polygon belongs to precisely two polygons. Suppose the cells of each dimension $\nu=0, 1, 2$, to be enumerated and let $\alpha^{(\nu)}_{i(\nu)}$, $i(\nu)=1, 2, \cdots, \alpha^{(\nu)}$, denote one of them. Let $\alpha^{(\nu,\mu)}_{i(\nu)}$ denote the number of μ-cells meeting this $\alpha^{(\nu)}_{i(\nu)}$ $(\mu\neq\nu)$. For this same cell, let $\alpha^{(\nu,\mu)}_{i(\nu)}$ denote the number of ν-cells which are incident with a μ-cell which is incident with the given $\alpha^{(\nu)}_{i(\nu)}$ (it is not stated whether a cell is to be counted more than once, or whether the cell $\alpha^{(\nu)}_{i(\nu)}$ itself is to be counted). Now let $\alpha^{(\nu,\mu)}=\sum_{i(\nu)}\alpha^{(\nu,\mu)}_{i(\nu)}$. The author asserts a number of identities, among them

$$\alpha^{(\nu,\mu)}=2\alpha^{(\mu)}, \qquad \nu\neq\mu,$$
$$\alpha^{(0,0)}-\alpha^{(1,1)}+\alpha^{(2,2)}=0,$$

from which there may be derived the following theorems [there is no indication of proof]. (1) In order that a given representation of R by simple polygons satisfy property (α) above, it is necessary and sufficient that the number ot polygons containing a common edge be a constant and tha- $\alpha^{(\nu,\mu)}=2\alpha^{(\mu)}$ $(\nu\neq\mu)$. (2) The polygonal decomposition satisfyf ing (α) will be a triangulation if and only if $\sum[\alpha^{(2,0)}_{i(2)}]^2=6\alpha^{(0)}$. (3) If R is a closed surface and N is its Euler characteristic then $N=\sum(\alpha_i^{(0,2)}-1)^2+\sum(\alpha_i^{(2,0)}-1)^2-\alpha^{(1,1)}$. (4) If, moreover, the decomposition is a triangulation: $3N=3\alpha^{(0)}-\alpha^{(1)}$. (5) If $\alpha^{(2)}>12$, then $\alpha^{(2,2)}>5\alpha^{(2)}$ and if in addition $N\leqq 0$, then $\alpha^{(2,2)}\geqq 6\alpha^{(2)}$. (6) Suppose that R is covered by a finite number, exceeding 12, of sets which are the closures of simply connected mutually exclusive domains, and suppose that the intersection of every two closures is vacuous or is connected. Then the arithmetic mean of the number of the closed sets which have a point in common with some one of these sets is greater than 5.

The author indicates extensions of (6) in which the condition of simple-connectedness may be dropped when the sets are of small enough diameter in the polyhedron. There is a reference to work of A. D. Alexandrov [Rec. Math. [Mat. Sbornik] (N.S.) 2(44), 307–318 (1937)] containing the "theorem of six neighbors" from which some of the results of this paper may be obtained. *L. Zippin.*

GEOMETRY

*Delachet, André. **La Géométrie Contemporaine.** Presses Universitaires de France, Paris, 1950. 128 pp.

This entertaining little book opens with a sketch of the history of geometry from Descartes to Cartan. This is followed by an elementary account of the geometries invariant under various groups of transformations, in the spirit of Klein [Vergleichende Betrachtungen über neuere geometrische Forschungen, Erlangen, 1872]. The author makes a nice distinction between Cartesian space (where a point is specified by three Cartesian coordinates), Arguesian space (where the coordinates are rendered homogeneous by the adjunction of the plane at infinity), and projective space (where this plane is not distinguished from any other). Turning to algebraic geometry, the study of properties invariant under birational transformations, he cites inversion in a circle as an instance of such a transformation. In an interesting digression he proves that an infinite set of points whose mutual distances are integers must be all on one line. [Cf. Anning and Erdős, Bull. Amer. Math. Soc. 51, 598–600 (1945); Erdős, ibid., 996 (1945); these Rev. 7, 164.] Topology is introduced by a long and excellent quotation from Poincaré [Dernières Pensées, Flammarion, Paris, 1913]. Euler's formula is attributed to Descartes [without any sufficient reason]. There is a discussion of the four-color problem, including a simple proof by Fréchet and Ky Fan that no finite number of colors can suffice for three-dimensional regions. The subject is brought up to date by mentioning the work of many living mathematicians, such as Whitney, Hodge, Freudenthal, Stiefel, and Ehresmann [whose name is spelt two different ways, both wrong]. *H. S. M. Coxeter* (Toronto, Ont.).

*Schläfli, Ludwig. **Gesammelte mathematische Abhandlungen. Band I.** Verlag Birkhäuser, Basel, 1950. 392 pp. (1 plate). 54 Swiss francs.

The foundations for hyperbolic geometry of two and three dimensions were laid by Gauss and Bolyai. Lobačevsky not only rediscovered the foundations but built up a vast structure of geometrical and trigonometrical results. Similarly Schläfli, besides rediscovering the foundations of n-dimensional Euclidean and spherical geometry, which had been laid by Grassmann and Möbius, developed the subject (about 1850) in a manner that was not approached by anyone else for half a century. The elegant volume under review, profusely annotated by Burckhardt, Hadwiger and Kollros, is a fitting tribute to the genius of this great Swiss mathematician (1814–1895). It begins with his early papers on solid geometry and special functions, including a thorough treatment of a problem that has lately aroused fresh interest: to find the largest ellipse touching four given lines, and the smallest ellipse passing through four given points [pp. 86–113]. These short papers are followed by a longer one, of some historical importance, on Jacobian elliptic functions.

On p. 167 we come to the pièce de résistance: the treatise in three parts entitled Theorie der vielfachen Kontinuität. The theory is based on n-dimensional Cartesian coordinates, but the geometrical objects soon reveal a degree of symmetry that transcends the coordinate system. Schläfli defines a polytope (Polyschem) as the set of points whose coordinates satisfy a system of linear inequalities, such that the content $\int^n dxdydz\cdots$ is finite. He immediately derives the generalized Euler formula $\sum_0^n(-1)^r a_r=1$, where a_r is the number of r-dimensional elements. In an interesting digression he obtains the expression $\sum_0^n\binom{i}{r}$ for the number of regions into which the Euclidean n-space is decomposed by i hyperplanes of general position. He uses the symbol (m, n) for the regular polyhedron whose faces are m-gons, n at each vertex, and (m, n, p) for the analogous four-dimensional polytope whose cells are (m, n), p at each edge. He determines the possible values of m, n, p by the inequality $\sin(\pi/m)\sin(\pi/p)>\cos(\pi/n)$, and tabulates the 600 tetrahedral cells of (3, 3, 5). He gives coordinates for the vertices of all the convex regular polytopes in any number of dimensions, and obtains their numerical and metrical properties. This ends part I. Part II begins with the content of the sphere in Euclidean n-space (in terms of the gamma function), and the content of the spherical simplex (Plagioschem) which is cut out by n hyperplanes through the center. This is, perhaps, the most ingenious part of the whole work. The content, being a transcendental function of the angles between the hyperplanes, is expressed by means of a differential equation. The case when $n=4$ was rediscovered fifty years later by Richmond [Quart. J. Pure Appl. Math. 34, 175–177 (1902); see also Coxeter, Quart. J. Math., Oxford

Ser. 6, 13–29 (1935)]. Schläfli established a remarkable relation between the contents of the $n+2$ spherical simplexes determined by a cycle of $n+2$ concurrent hyperplanes, each perpendicular to all except its two neighbors. The 3-dimensional case is the pentagramma mirificum of Napier and Gauss; the 4-dimensional case was rediscovered by Wythoff [Akad. Wetensch. Amsterdam. Proc. Sect. Sci. 9, 529–534 (1907); see also Coxeter, Bull. Calcutta Math. Soc. 28, 123–144 (1936)]. Part II closes with a discussion of eutaxy: a set of vectors is said to be eutactic if the sum of the squares of their orthogonal projections on a line is the same in all directions. This idea has borne fruit in some recent work of Hadwiger [Comment. Math. Helv. 13, 90–107 (1940); these Rev. 2, 260; see also Coxeter, Regular Polytopes, Pitman, New York, 1949; these Rev. 10, 261]. Part III deals with n-dimensional quadrics: conjugate diameters, confocals, etc., ending with some results in n-dimensional potential theory. *H. S. M. Coxeter.*

Bilo, Julien. Contribution to the study of the foundations of ordinary complex projective geometry and to the purely synthetic study of the complex one-dimensional primitive forms. Verh. Vlaamsche Acad. Kl. Wetensch. 11, no. 29, 152 pp. (1949). (Dutch. French summary)

This memoir provides an axiomatic introduction to complex projective geometry, expanding and slightly modifying the treatment of Veblen and Young [Projective Geometry, v. 2, Boston, 1918, pp. 16–17, 21–23, 30–33]. A chain is derived from a harmonic net by adjoining its limit points (Dedekind cuts). The departure from real geometry is secured by an axiom to the effect that a chain does not exhaust the whole line. An elementary projectivity

$$\begin{Bmatrix} X & Y & A \\ X & Y & A_1 \end{Bmatrix}$$

on a line r is defined as the product of two perspectivities

$$r \overset{S}{\wedge} p \overset{S_1}{\wedge} r,$$

where the line SS_1 passes through X, and p through Y. This projectivity is said to be parabolic if X and Y coincide, so that there is only one invariant point. Two chains on r are said to be tangent at X if they are related by such an elementary parabolic projectivity. A rather complicated axiom on chains enables the author to prove Pappus's theorem (and the fundamental theorem) by identifying the two nonparabolic projectivities

$$\begin{Bmatrix} X & Y & A \\ X & Y & A_1 \end{Bmatrix} \text{ and } \begin{Bmatrix} Y & X & A \\ Y & X & A_1 \end{Bmatrix}.$$

When such a projectivity is involutory, it is called a harmonic correspondence. Besides its two invariant points it has an infinite number of invariant chains. It is determined by one of its invariant points, say X, and any invariant chain k. The other invariant point Y is said to be symmetric to X with respect to k, or to be derived from X by reflection in k. The reflection in a chain is an involutory antiprojectivity: it transforms chains into chains but is not a product of perspectivities. Any antiprojectivity may be expressed as the product of a suitable projectivity with the reflection in an arbitrarily chosen chain. All these and many other results are obtained synthetically, without appealing to the Argand diagram wherein chains and reflections are represented by circles and inversions. *H. S. M. Coxeter.*

***Bilo, Julien. Onderzoekingen betreffende de Meetkundige Grondslagen van de Projectieve Quaternionenmeetkunde. [Investigations on the Geometrical Foundations of Projective Quaternion Geometry].** A. Vanderlinden, Brussel, 1949. 123 pp. 75 Belgian francs.

In his book on complex projective geometry [see the preceding review] the author used such a set of axioms that only the final one has to be denied when the field is extended from complex numbers to quaternions. The chain defined there is now called a 1-chain (literally "chain of the first kind") because other kinds come in as well. An i-chain on the quaternion line ($i=2$, 3 or 4) is determined by an $(i-1)$-chain s, a point X on s, and a point S not on s; it consists of the points of all 1-chains through S and X that meet s once more or else touch s. The existence of such chains is postulated for $i=1$, 2, 4 (and deduced for $i=3$), and an axiom of closure states that the whole line is a 4-chain. Since quaternions are not commutative, the specification of a projectivity requires a statement of the transforms of at least four points. An elementary projectivity

$$\begin{Bmatrix} X & Y & A & B \\ X & Y & B & C \end{Bmatrix}$$

that differs from

$$\begin{Bmatrix} Y & X & A & B \\ Y & X & B & C \end{Bmatrix}$$

is said to be nonhyperbolic. The first "fundamental theorem" asserts that a projectivity which has as many as three invariant points is either the identity or the reflection in a 2-chain. This kind of reflection is expressible as the product of two quasi-inverse nonhyperbolic projectivities

$$\begin{Bmatrix} Y & X & B & C \\ Y & X & A & B \end{Bmatrix} \begin{Bmatrix} X & Y & A & B \\ X & Y & B & C \end{Bmatrix}.$$

The second "fundamental theorem" asserts that every chain-preserving transformation of a line into itself is either a projectivity or an antiprojectivity: the product of a projectivity and the reflection in a 1-chain. The reflection in an i-chain is a projectivity or an antiprojectivity according as i is even or odd. Every involutory projectivity is either a harmonic correspondence or the reflection in a 2-chain. Every involutory antiprojectivity is either the reflection in a 1-chain or the reflection in a 3-chain or an elliptic antiinvolution: the product of the reflection in a 3-chain and a harmonic correspondence whose invariant points are images of each other by that reflection.

The book is well written, beautifully printed, and equipped with an adequate bibliography. But it has no index and (perhaps inevitably) no illustrations. *H. S. M. Coxeter.*

Godeaux, Lucien. Sur une propriété des quadriques. Simon Stevin 27, 65–68 (1950).

Let Q be a quadric, a and b two tangents to Q such that the plane joining a and the tangent point of b is conjugate with the plane joining b and the tangent point of a. The author determines and investigates the locus of those tangents to Q which intersect a and b. This locus is composed of two quadrics. *W. van der Kulk* (Providence, R. I.).

Sen Gupta, B. K. Note on the closest contact of a conic. Math. Student 17 (1949), 38–40 (1950).

Venkataraman, M. The Langley chain. Math. Student 17 (1949), 36–37 (1950).

Thébault, Victor. Sphères remarquables associées à un tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. 63, 130–136 (1949).

Visser, D. An approximate construction for the trisection of an angle. Simon Stevin 27, 69–70 (1950). (Dutch)

de Jong, B. J. On the trisection of an angle. Simon Stevin 27, 71–92 (1950). (Dutch)

The author gives a more than usually careful proof of the impossibility of a ruler and compass trisection, describes some exact trisections, and discusses several approximate ruler and compass trisections. *R. P. Boas, Jr.*

Algebraic Geometry

Anzellotti, Adriana. Sulla classificazione delle famiglie di curve algebriche sghembe, mediante gli n-lateri in esse contenuti. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 8, 309–327 (1949).

Severi [Vorlesungen über algebraische Geometrie, Teubner, Leipzig-Berlin, 1921, Anhang G] showed that curves of order n and genus p in space of r dimensions form for $n \geqq p+r$ a single irreducible family which includes all n-laterals of this genus, and for $r+rp/(r+1) \leqq n < p+r$ a finite system of irreducible families, each of which includes some n-laterals of genus p. (An n-lateral of genus p is a set of n distinct lines with a total of $n+p-1$ simple intersections by pairs, which make the whole set connected, i.e., however the whole set is divided into two subsets there is at least one intersection of lines which are one in each subset.) Limiting herself to $r=3$ the author proves that every complete irreducible family of curves of order n and genus p defined as the intersection of two surfaces of orders m_1, m_2, residual to a curve C' of order n' and genus p' contains some n-laterals of genus p provided either (i) one of $m_1, m_2 = 2$; or (ii) $n'=0$ or 1; or (iii) C' belongs to a family containing an n-lateral C_0' of genus p' such that the general surfaces of orders m_1, m_2 through it are irreducible, and on a surface of order m_1 through C_0' the surfaces of order m_2 trace, residually to C_0', a linear system of freedom greater than 2 and without fixed parts.

These results are applied to determine whether known families of curves not satisfying Severi's inequality above nevertheless contain n-laterals. Halphen [J. École Polytech. Cahier 52, 1–200 (1882)] listed all families for $n \leqq 12$, over 100 in all. It is found that all of these families, of which 22 fail to satisfy Severi's inequalities, nevertheless contain n-laterals of the right genus, with the possible exceptions of the cases $n=12$, $p=13, 14$, to which the tests found above could not be applied. Examples are then given of n-laterals of genus p, where $n < p+3$, which are not contained in any continuous family of irreducible curves of the same order and genus; either because $p > (n-2)^2/4$ so that there is no irreducible twisted curve of this order and genus, or because every irreducible curve of this order and genus lies on a quadric while the n-lateral does not. *P. Du Val.*

Pissard, Nelly. Sur les surfaces à sections hyperplanes hyperelliptiques. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) 23, no. 1, 35 pp. (1949).

Ce mémoire a pour but d'étendre aux surfaces à sections hyperelliptiques de genre π quelconque certaines propriétés déjà obtenues par l'auteur dans le cas $\pi=3$ [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 859–865 (1942); ces Rev. 7, 73]. La première partie du mémoire est consacrée à certaines surfaces rationnelles normales de S_r définies par leur représentation plane, situées sur la section par S_r d'une variété de Segre W_{r+1}^{r+1}. L'auteur en déduit dans la seconde partie que toute surface rationnelle normale $F^{4\pi+4}$ à sections hyperelliptiques de genre π appartient à une $V_3^{2\pi+3}$ lieu de plans qui la coupent suivant des coniques. La surface $F^{4\pi+4}$ est l'intersection de $V_3^{2\pi+3}$ et d'une hyperquadrique qui ont en outre $2\pi+2$ plans communs; $V_3^{2\pi+3}$ est soit la section par $S_{3\pi+5}$ d'une variété $W_{3\pi+3}^{3\pi+3}$ de Segre, soit un cône projetant d'un point une réglée rationnelle normale $R^{2\pi+3}$ admettant une directrice d'ordre $\pi+1$ (dans ce dernier cas les plans résiduels appartiennent à un même hyperplan).

La troisième partie du mémoire est consacrée à une étude plus particulière de la surface F_0 représentée par le système complet des C^n ayant un point multiple A donné d'ordre $n-2$, et aux projections de cette surface faites à partir des tangentes en h points aux coniques de F_0 qui y passent. Cette étude généralise une étude antérieure de Rozet [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 23, 356–368 (1936)]. La dernière partie est une application de la précédente à l'étude de l'existence des courbes planes C^n ayant un point A multiple d'ordre $n-4$ et h autocontacts à tangentes passant par A: on peut assurer que ces courbes existent lorsque $5n \geqq 6(h+1)$. Pour $h=n-4$ ce résultat est conforme à une conjecture de Enriques [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 7_2, 281–286, 344–347 (1898)].

L. Gauthier (Nancy).

Legrain-Pissard, [N.]. Sur une transformation birationnelle involutive de Jonquières de l'espace. Bull. Soc. Roy. Sci. Liége 18, 357–362 (1949).

Oleĭnik, O. A. On the topology of real algebraic space curves. Doklady Akad. Nauk SSSR (N.S.) 70, 13–14 (1950). (Russian)

Soient Γ et γ deux surfaces algébriques réelles d'ordres respectifs p et q dans l'espace projectif réel à trois dimensions, ces surfaces se coupant suivant une courbe réelle K. Désignons par M_0 la multiplicité obtenue à partir de Γ en supprimant la courbe K lorsque q est pair, et en supprimant à la fois la courbe K et une section plane, si q est impair. En désignant par $E(M)$ la caractéristique d'Euler-Poincaré de la multiplicité M, l'auteur indique un lemme, application d'un résultat de Petrovskiĭ sur la topologie des courbes algébriques planes réelles [Ann. of Math. (2) 39, 189–209 (1938)] et de son extension aux surfaces algébriques réelles par Petrovskiĭ et l'auteur [mêmes Doklady (N.S.) 67, 31–32 (1949); ces Rev. 11, 204] qui permet d'aboutir au théorème suivant:

$$|E(M_0)| \leqq \tfrac{1}{3}p^3+\tfrac{1}{2}pq^2+\tfrac{1}{4}p^2q+\tfrac{1}{2}|E(\Gamma)|+m$$

où $m=-p^2-pq+\frac{7}{4}p$ si q est pair, $m=-\frac{3}{4}p^2-\frac{1}{4}pq+13p/24$ si q est impair. La démonstration est seulement esquissée.

L. Gauthier (Nancy).

Petrovskiĭ, I. G., and Oleĭnik, O. A. On the topology of real algebraic surfaces. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 389–402 (1949). (Russian)

In this paper the authors give the proofs of some results which they announced in a preceding paper [Doklady Akad. Nauk SSSR (N.S.) 67, 31–32 (1949); these Rev. 11, 204]. These results yield upper bound estimates, depending only on m and n, for the absolute value of the Euler-Poincaré

characteristic (mod 2) $E(\Gamma_0)$ of any real hypersurface Γ_0 of order n in the projective m-space S_m, under the assumption that Γ_0 is free from real singularities. If $F(x_1, \cdots, x_m)=0$ is the equation of Γ_0 and if M_0 denotes the closure, in S_m, of the set of points at finite distance where $F \geqq 0$, then similar estimates are obtained also for $E(M_0)$. An important preparatory step consists in replacing Γ_0 by a very near hypersurface Γ_0' which is sufficiently general for all the purposes of the proof. This is possible since it is proved that Γ_0' and the corresponding set M_0' are homeomorphic to Γ_0 and M_0 respectively. With this preparation of Γ_0 it is then permissible to assume that the "critical points" of the polynomial $F(x)$, i.e., the intersections (real or imaginary) P_α of the m polar hypersurfaces $F_i'=0$ are exactly $(n-1)^m$ in number and that F takes distinct nonzero values at these points. These assumptions imply that if Γ_C denotes the hypersurface $F-C=0$ (C a real constant) and M_C is the analogue of M_0 for Γ_C, then $E(M_C)$ increases or diminishes by 1 as C passes through a critical value $F(P_\alpha)$, P_α real. By letting the real parameter C vary between two fixed constants C_1 and C_2 such that $C_2<F(P_\alpha)<C_1$ for all real critical points P_α, the authors obtain the relation

$$E(M_0)-\tfrac{1}{2}[E(M_{C_1})+E(M_{C_2})]=A+B'+(n-1)^m/2-\gamma,$$

where the symbols have the following significance: (1) A is the number of real critical points P_α such that $0<F(P_\alpha)<C_1$ and such that the (nonsingular) quadratic form which approximates, locally at P_α, the polynomial $F(x)-F(P_\alpha)$ has signature $m-2k$, k even; (2) B' has a similar significance for C_2, but k now must be odd; (3) 2γ is the number of imaginary critical points P_α. An algebraic lemma and an interpolation formula of Euler-Jacobi enable the authors to establish the crucial inequality $A+B'>S(m, n)-\gamma-1$, where $S(m, n)$ is the number of linearly independent polynomials in $x_1, x_2, \cdots, x_m$ which are of degree at most $n-2$ in each of the variables x_i and of total degree at most $[(mn-2m-n)/2]$. This inequality, combined with the preceding result, yields the estimate

$$|E(M_0)-\tfrac{1}{2}[E(M_{C_1})+E(M_{C_2})]|<(n-1)^m/2-S(m, n)+1.$$

Using the well-known formula

$$E(K_1+K_2)=E(K_1)+E(K_2)-E(K_1K_2)$$

for polyhedra K_1 and K_2 it is then proved that

$$E(M_{C_1})+E(M_{C_2})=1$$

if n is even. This, in conjunction with the relation

$$E(M_0)+E(S_m-M_0)=E(S_m)+E(\Gamma_0),$$

yields an upper bound for $|E(\Gamma_0)|$ when n is even. The case of n odd is reduced to the case of an even n by considering the section of Γ_0 with the hyperplane at infinity.

O. Zariski (Cambridge, Mass.).

Mathieu, Paulette. Extension à l'espace à cinq dimensions de la correspondance involutive de Reye. C. R. Acad. Sci. Paris **230**, 1132–1134 (1950).

These two pages are themselves such a closely packed summary of results that it is not easy to summarize them further. The work is a detailed study of an involution I in S_5, determined by quadrics through a canonical curve C^{10} of genus 6, which are an ∞^5 linear system such that all of them that pass through a general point M pass also through a unique point M', companion of M in I. Fundamental elements are the five W_3^{30}'s, generated by the planes of the five pencils of conics on the unique del Pezzo surface F^5 of which C^{10} is a quadric section, and the variety V_3^{30} of chords of C^{10}. The united (Jacobian) locus is sextic U_4^6, containing the five W_3^{30}'s and V_3^{30} simply, F^5 doubly and C^{10} triply. The projective model of the quadrics through C^{10} is a double S_5 whose branch locus, image of U_4^6 (in the same way as the Kummer surface is of the Weddle surface) is $D_4'^6$; this has a quadruple point O', image of F^5, whose tangent cone has 15 double planes (projecting the familiar figure of 15 lines in S_4, six associated sets of five). One associated set of five of these planes are images of the five W_3^{30}'s, the other ten are images of the ten lines on F^5. The former five planes are double on $D_4'^6$, as is the surface F'^{10}, image of V_3^{30}; this has a sextuple point at O'. Many other properties are given.

P. Du Val (Athens, Ga.).

Castelnuovo, Guido. Sul numero dei moduli di una superficie irregolare. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 3–7 (1949).

Castelnuovo, Guido. Sul numero dei moduli di una superficie irregolare. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 8–11 (1949).

L'auteur établit sous la condition qu'une surface algébrique soit irrégulière et dépourvue d'un faisceau irrationnel de courbes, l'inégalité $M \leqq 3p_g-2p_a$ pour le nombre de ses modules. Partant d'un système linéaire $|C|$ assez ample pour contenir le système bicanonique $|2K|$ et régulier, il montre que le nombre des courbes linéairement indépendantes du système $|2C+2K|$ passant par le groupe jacobien G_δ d'un faisceau extrait de $|C|$ est l'invariant θ qui apparait dans la formule donnée par Enriques pour le nombre des modules [Le superficie algebriche, Zanichelli, Bologna, 1949, pp. 204 ff.; ces Rev. 11, 202]. Considérant alors les courbes de $|2C+2K|$ passant par G_δ et une X de $|2C+K|$ y passant, courbes qui avec les hypothèses existent et sont irréductibles, il montre par voie transcendante que $(K, X)=G_\gamma+G_{\delta+\epsilon}$, $G_{\delta+\epsilon}$ étant le groupe des points doubles d'un faisceau de $|C|$ et G_γ le résidu de $G_\delta+G_\epsilon$ (G_ϵ groupe des points-base du faisceau) par rapport à un groupe caractéristique de X. Sur X, la série g_δ définie par G_δ contient $G_{\delta+\epsilon}$ si sont vérifiées $\eta+1$ conditions. Les caractères des séries utilisées donnent $\eta \leqq M \leqq \eta+p_g-p_a$. Utilisant alors la série $g_{\delta+\epsilon}$ déterminée par G_δ et G_ϵ pour laquelle $G_{\delta+\epsilon}$ impose $\eta'+1$ conditions on trouve $\eta \leqq \eta'=\omega-1$, ω étant un invariant défini par le nombre $1+4-\omega$ des conditions qu'impose le $G_{\delta+\epsilon}$ aux courbes bicanoniques y passant. Mais sur une bicanonique ce groupe appartient à une série de dimension ρ inférieure à p_g+1 si l'irrégularité $\geqq 3$, d'où $M \leqq 3p_g-2p_a$. Un exemple laisse à penser que cette inégalité ne puisse être facilement améliorée.

B. d'Orgeval.

Godeaux, Lucien. Remarques sur les surfaces algébriques irrégulieres. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 464–469 (1949).

Alcune osservazioni sopra le superficie algebriche di irregolarità $q>0$, sulle quali esistano curve di genere $\pi<q$. Si dimostra che se una superficie F d'irregolarità q contiene una curva K di genere $\pi<q$, la K è isolata o appartiene ad un fascio irrazionale. In quest'ultima eventualità, si stabiliscono alcuni rapporti, che l'autore si propone di approfondire in altri lavori, tra la varietà di Picard della F, la varietà di Jacobi della K e la varietà di Jacobi della curva rappresentativa del fascio irrazionale, in ordine all'esistenza su tali varietà abeliane di sistemi di imprimitività.

F. Conforto (Roma).

Andreotti, A. Le serie lineari sopra una retta multipla ed in particolare sopra una retta doppia. Ann. Mat. Pura Appl. (4) **27**, 165–175 (1948).

Sia data una curva C di genere $p>0$, rappresentata mediante una g_ν^1 sopra una retta ν-pla χ; le serie lineari g_n^r di C si mutano in serie ∞^r γ_n^r d'indice ν' su χ (a meno che g_n^r sia composta con la g_ν^1). Per caratterizzare le serie γ_n^r, l'autore osserva che la corrispondenza $(\nu, 1)$ fra C e χ, induce tra la varietà W_n delle n-ple di punti di C e quella S_n delle n-ple di punti di χ una corrispondenza $(\nu^n, 1)$ tale che, assumendo come modello proiettivo di S_n una spazio lineare d'iperpiani, le varietà W_r immagini delle g_n^r di C si mutano in sistemi ∞^r di iperpiani di S_n, i quali segano sulla immagine proiettiva Γ di χ la corrispondente serie γ_n^r. L'autore prova che (tolto il caso in cui ogni gruppo di γ_n^r provenga da più gruppi di g_n^r) un tale sistema di iperpiani è costituito dagli iperpiani tangenti ad una forma Φ (luogo di punti) di S_n formata di ∞^r S_{n-r-1} lungo i quali l'iperpiano tangente è fisso.

Il problema della caratterizzazione delle serie γ_n^r è così ricondotto a quello delle forme Φ. Nel caso $\nu=2$ delle rette doppie, l'esame è portato a fondo. Pel caso delle serie complete e non speciali, l'autore prova il seguente teorema: rappresentata una retta doppia χ sopra una curva razionale normale Γ di S_n, le ∞^p serie γ_n^r $(n=r+p)$ sono segate su Γ dagli iperpiani tangenti alle forme d'ordine $r+1$ di S_n che passano r volte pei punti di diramazione, $r-1$ volte per Γ e che si ottengono: se $p\leqq 2r-1$ come sezioni con uno S_n della forma ψ di S_{ρ_r} $(\rho_r=r(r+3)/2)$ luogo degli ∞^r $S_{\rho_{r-1}}$ di appartenenza delle V_{r-1}^{2r-1} di Veronese contenute in una varietà analoga V_r^{2r} di dimensione superiore d'una unità; se $p>2r-1$ come proiezioni da uno S_{p-2r} di una sezione con uno S_{2r-1} della stessa forma ψ. Anche il caso delle serie speciali e quello delle serie incomplete vengono esaminati. Caso particolare interessante è quello delle rette doppie ellittiche, che viene anche trattato direttamente. Tutto ciò getta luce sopra una nota equazione differenziale di Eulero e su certi sistemi abeliani, che la generalizzano.

F. Conforto (Roma).

Chow, Wei Liang. Algebraic systems of positive cycles in an algebraic variety. Amer. J. Math. **72**, 247–283 (1950).

Soit U/K une variété; à tout cycle positif G de U on fait correspondre les coéfficients de sa "forme associée" [Math. Ann. **113**, 692–704 (1937)], c'est-à-dire un point d'un certain espace projectif S_t. Aux cycles G de degré et de dimension donnés correspondent les points d'un ensemble algébrique de S_t. Toute sous variété V/K de celui-ci est appelée un système algébrique de cycles positifs sur U. L'auteur suppose que le cycle générique $G(y)$ de V est un cycle premier rationnel sur $K(y)$, dont un point générique (x) est point générique de U/K (ceci revient à dire que U est le "porteur" du système V); alors la variété T de point générique (y, x) sur K est appelée la correspondance associée à V (elle fait correspondre à un point de U les cycles de V qui passent par ce point). Lorsque T est une transformation rationnelle de U sur V (c'est-à-dire si, par un point générique de U passe un cycle et un seul de V), le système V est dit "involutionnel," les "involutions" classiques en étant le cas particulier où dim $(G(y))=0$. Dans ces conditions le principal résultat démontré est le suivant: (y') étant un point de V, (x') un point générique d'une composante simple G'/K de $G(y')$, et T étant régulière en (x'), alors (sous certaines conditions de séparabilité et de disjonction analytique) les variétés U et $V\times G'$ sont analytiquement équivalentes (c'est-à-dire ont des anneaux locaux complétés isomorphes) aux points (x') et (y', x'). Outre une application à la théorie des Jacobiennes, ce théorème de "décomposition locale en produit" est un pas vers une théorie des espaces fibrés en géométrie algébrique.

Avant de démontrer ce théorème, l'auteur précise la terminologie employée, qui se rapproche de celle de A. Weil, tout en insistant un peu plus sur les corps de base et sur les variétés projectives. Deux lemmes donnent des propriétés des anneaux locaux liés à la situation géométrique ci-dessus: si U et V ont même dimension $Q(V, y')$ est un sous anneau et un sous espace de $Q(U, x')$ [cf. Zariski, Proc. Nat. Acad. Sci. U. S. A. **35**, 62–66 (1949); ces Rev. **10**, 396]; l'autre est une propriété due au choix de la forme associée pour représenter les cycles. La démonstration du théorème se fait alors par réduction au cas où dim $(G)=0$. Les théorèmes généraux sur les anneaux locaux, utilisés ici, sont donnés en appendice: il y est surtout question de domaines locaux A (anneaux locaux sans diviseurs de zero) et de leurs extensions A_L par extension en L du corps de base K (obtenues par formation d'un produit tensoriel et d'un anneau de fractions); la notion principale est la suivante: L est dit analytiquement disjoint de A sur K, si, dans le complété $(A_L)^*$ de A_L, les idéaux $(A_L)^*P_i$ sont premiers, les P_i étant les idéaux premiers isolés de (0) dans A^*.

P. Samuel.

Northcott, D. G. Periodic points on an algebraic variety. Ann. of Math. (2) **51**, 167–177 (1950).

Let F and R be any field of complex numbers and the rational field, respectively. Denote then by σ any relative isomorphism of F/R, and by ξ^σ its effect on a given element ξ of F. A point $P=(\xi_0, \cdots, \xi_n)$ of the linear space $S_n(x_0, \cdots, x_n)$ is said to be rational with respect to F if its homogeneous coordinates ξ_i are equal (or proportional) to elements of F. The degree of rationality of a point P of S_n, whose coordinates are algebraic numbers, is the degree over R of the smallest field in which P is rational. The author proves first that, given a positive number k, for any field F there are only a finite number of points $P=(\xi_0, \cdots, \xi_n)$ of S_n which are rational relative to F and such that $D_F(P)\leqq k$, where $D_F(P)=\prod_\sigma(|\xi_0^\sigma|+\cdots+|\xi_n^\sigma|)/(N_{F/R}\Omega)$, the product being taken over all different isomorphisms σ of F/R, and Ω denoting the ideal sum $(\xi_0)+\cdots+(\xi_n)$.

Consider next an algebraic variety V of S_n, and the mapping of S_n carrying the point $(x_0, \cdots, x_n)$ into

$$[L_0(x_0, \cdots, x_n), L_1(x_0, \cdots, x_n), \cdots, L_n(x_0, \cdots, x_n)],$$

where the $L_i(x)$ are forms of degree l having algebraic numbers as coefficients. Suppose that (i) $l\geqq 2$, (ii) the $L_i(x)$ never vanish simultaneously at any point of V, (iii) the mapping transforms V into itself; then it is proved that the number of points of V having a given degree of rationality and such that, by iterating the mapping, each of them generates a finite sequence, is always finite. This is obtained as a consequence of the inequality $D_F{}^l(P)\leqq M^{(F:R)}D_F(Q)$, where P is any point of V which is rational relative to F, Q is the transform of P by the mapping, and M is a constant which depends on V and the mapping, but not on F. In the particular case where V is an elliptic cubic curve, the above results are applied to the study of the division values of the Weierstrass $\wp$-function, thus extending results previously obtained in this connection by K. Mahler [Quart. J. Math., Oxford Ser. **6**, 74–77 (1935)].

Conditions (i)–(iii) above are very restrictive [cf. B. Segre, Math. Ann. **109**, 1–3 (1933); F. Severi, Math. Ann.

109, 4–6 (1933); these papers, however, are not quoted by the author]. The author succeeds in weakening condition (ii) for the case of plane curves, by utilizing Weil's concept of a "distribution" and his "decomposition theorem" [Acta Math. 52, 281–315 (1929)]. *B. Segre* (Bologna).

Differential Geometry

Blaschke, Wilhelm. Contributi alla cinematica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 268–280 (1949).

This paper reproduces two lectures, concerned with a variety of problems in plane kinematics. It is first shown that Lie's contact transformation which transforms the straight lines in three-dimensional space into spheres can be represented as the product of two simpler transformations, an isotropic projection and the inverse of a kinematic projection. The discussion is then turned to two-parameter families of rigid motions of one plane upon another. Such a family can be represented, in an infinity of ways, by a surface in three-dimensional space, the surfaces representing the same family being equivalent under the operations of a three-parameter subgroup of Lie's group of contact transformations carrying spheres into spheres. Various properties of the families of motions, particularly some properties involving the concepts of integral geometry, are discussed in terms of this representation. The paper closes with some brief remarks concerning the relations between the kinematics on a sphere and the geometry of Euclidean 3-space.

L. A. MacColl (New York, N. Y.).

Süss, Wilhelm. Bestimmung einer Fläche durch die dritte Grundform und die Summe der Hauptkrümmungsradien. Arch. Math. **2**, 103–104 (1950).

A surface in ordinary space is uniquely determined if there are given: a strip, and (as functions of the parameters) the third fundamental form, i.e., the first fundamental form of the spherical image, and the sum of the principal radii of curvature [W. Scherrer, Comment. Math. Helv. **20**, 366–381 (1947); these Rev. **9**, 464]. The author proves this by first determining the distribution of the parameters on the unit sphere. *W. Fenchel* (Princeton, N. J.).

Sorace, O. Nuclei risolventi di Volterra relativi ad un gruppo di curve. Boll. Accad. Gioenia Sci. Nat. Catania (4) no. **1**, 59–73 (1948).

Given both curvatures of a curve C in a three-space, C may be found by means of a Volterra integral equation [P. Nalli, Ann. Mat. Pura Appl. (4) **17**, 193–202 (1938)]. The author shows that the resolvent kernel for this equation may be obtained in finite terms for a helix and its two first evolutes. *V. Hlavatý* (Bloomington, Ind.).

Rollero, Aldo. Ancora sulla rappresentazione delle trasformazioni puntuali fra piani mediante sistemi di equazioni differenziali. Euclides, Madrid **9**, 375–378 (1949).

Let X_i $(i=1, 2, 3)$ be projective homogeneous coordinates of a point of a plane Π; and let x, y be projective non-homogeneous coordinates in another plane π; in a paper which has not yet appeared the author has proved that any point-to-point transformation T between Π and π may be represented by the following equations: $X_i=X_i(x, y)$, where the $X_i(x, y)$ are integrals of a system of partial differential equations of the 2d order:

$$(1)\qquad \begin{cases} X_{xx}=pX+\alpha X_x+\beta X_y, \\ X_{xy}=qX-\zeta X_x-\alpha X_y, \\ X_{yy}=rX+\eta X_x+\zeta X_y \end{cases}$$

($p, q, r, \alpha, \beta, \eta, \zeta$, functions of x, y). In the same paper, the author has shown also that, by a suitable choice of the non-homogeneous coordinates $\bar{X}, \bar{Y}, \bar{x}, \bar{y}$ in Π, π, T may be represented in the neighbourhood of the 2d order of two corresponding points P and P' in the following way:

$$(2)\qquad \begin{cases} \bar{X}=\bar{x}+\frac{1}{2}\alpha\bar{x}^2-\zeta\bar{x}\bar{y}+\frac{1}{2}\eta\bar{y}^2+[3], \\ \bar{Y}=\bar{y}+\frac{1}{2}\beta\bar{x}^2-\alpha\bar{x}\bar{y}+\frac{1}{2}\zeta\bar{y}^2+[3]. \end{cases}$$

In this paper the author determines the coordinate transformation that leads from expansion (2) to a canonical expansion for T, up to the 2d order, of two corresponding points P and P' found, in quite a different way, in another paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 213–216 (1949); these Rev. **11**, 210]. By calculating the expansion (2) up to the 3d order, it is possible to express for $p, q, r, \alpha, \beta, \eta, \zeta$ (and their first derivatives) the six projective invariants of T (up to the 3d order).

V. Dalla Volta (Princeton, N. J.).

Smirnov, R. V. Laplace transforms of p-conjugate systems. Doklady Akad. Nauk SSSR (N.S.) **71**, 437–439 (1950). (Russian)

A p-dimensional surface in n-dimensional projective space is called a p-conjugate system if it is possible to take on it p families of lines, satisfying the following conditions. (1) Through every point A_0 of the surface pass p lines of the net in p linearly independent directions; (2) the tangents to any line of any family taken along any line of any second family form a two-dimensional developable surface. Such a system is determined by the equation

$$(1)\qquad \omega^\alpha=0,\quad \omega_i^j=a_i^j\omega^i+b_i^j\omega^j,\quad \omega_i^\alpha=a_i^\alpha\omega^i,$$
$$i\neq j;\quad i, j=1, \cdots, p;\quad \alpha=p+1, \cdots, n \text{ (do not sum)}$$

with respect to a frame of reference consisting of A_0, points $A_1, \cdots, A_p$ on the tangents to the p lines passing through A_0 and belonging to different families and any general points $A_{p+1}, \cdots, A_n$ outside the tangent plane. The system (1) is in involution and determines the X_p with $p(p-1)$ arbitrary functions of two variables. The point $A_i^k=b_i^kA_0+A_i$ describes the edge of regression of the two-dimensional developable surface formed by the motion of the tangent to one of the lines of the family ω^i (that is, $\omega^1=\cdots=\omega^{i-1}=\omega^{i+1}=\cdots=\omega^n=0$) along a line of the family ω^k. The point A_i^k is called a focus; there are $p(p-1)$ foci. When A_0 moves in the p-conjugate system every point A_i^k describes a surface which in general is a p-conjugate system. This surface A_i^k is the Laplace transform of the surface A_0.

The author enumerates a number of properties of these systems. By repeating the process s times we obtain for the number of new p-conjugate systems:

$$N(s, p)=\sum_{l=0}^{s-1}\binom{s-1}{l}\binom{p}{l+1}\sum_{k=p}^{s-1}\binom{s-1}{k}\binom{p-l-1}{s-k},$$

$$\binom{k}{0}=\binom{k}{k}=1.$$

If we take on every ray A_0A_i $(i=1, \cdots, p)$ a point $B_i=\nu_iA_0+A_i$ such that

$$[\alpha B_iB_iB_j]=0,\quad \text{mod } (\omega^1, \cdots, \omega^{i-1}, \omega^{i+1}, \cdots, \omega^p),$$

then this point describes a p-conjugate system, and some

properties of the systems B_i are found. The case is also studied in which a q-conjugate system S_q is formed from an S_p by taking $\omega^{q+1}=\cdots=\omega^p=0$. These results are related to those of E. Cartan [Bull. Soc. Math. France **47**, 125–160 (1919); **48**, 132–208 (1920)], T. Kosmina [C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 183–185 (1947); these Rev. **8**, 531] and S. Chern [Proc. Nat. Acad. Sci. U. S. A. **30**, 95–97 (1944); these Rev. **5**, 217]. *D. J. Struik.*

Urban, A. On the geometry of a system of partial differential equations of the second order. Nederl. Akad. Wetensch., Proc. **52**, 855–867 = Indagationes Math. **11**, 303–315 (1949).

Consider a completely integrable system

$$\text{(1a)}\qquad \frac{\partial^2 z}{\partial\xi^\lambda\partial\xi^\mu}-\Gamma^{\nu}_{\lambda\mu}\frac{\partial z}{\partial\xi^\nu}=\Gamma_{\mu\lambda}z;$$

put (1b) $\xi^0\equiv-\log z$ and denote by X_{n+1} the space of the coordinates ξ^a $(a,b,c=0,\cdots,n)$ with allowable transformations

$$\text{(2)}\qquad \xi^{0\prime}=\xi^0-\log\rho(\xi^1,\cdots,\xi^n),\quad \xi^{\nu\prime}=\xi^{\nu\prime}(\xi^1,\cdots,\xi^n).$$

There is a connection in X_{n+1}, defined to within the usual projective transformation

$$\text{(3)}\qquad {}^*\Gamma^a_{bc}={}^0\Gamma^a_{bc}+p_b\delta^a_c+p_c\delta^a_b$$

such that every solution $\xi^0=f(\xi^1,\cdots,\xi^n)$ of (1) is a geodesic X_n in X_{n+1} with respect to it and vice versa. The vector p in (3) may be fixed by further geometrical requirements imposed either on these X_n or on the corresponding curvature tensor. On the other hand the transformation (2) suggests the existence of projective coefficients $\Pi^\nu_{\lambda\mu}$ in a sense of T. Y. Thomas. One obtains them by eliminating ρ from the transformation formulas for $\Gamma^\nu_{\lambda\mu}$. If $\Gamma_{\mu\lambda}$ is a tensor of rank n and $\{^\nu_{\lambda\mu}\}$ its corresponding Christoffel symbols, then

$$\frac{\partial^2\zeta}{\partial\xi^\mu\partial\xi^\lambda}-{}^1\Gamma^{\nu}_{\mu\lambda}\frac{\partial\zeta}{\partial\xi^\nu}=\Gamma_{\mu\lambda}\zeta$$

$$\left({}^1\Gamma^\nu_{\lambda\mu}=2\left\{{}^{\nu}_{\mu\lambda}\right\}-\Gamma^\nu_{\mu\lambda}\right)$$

is the Bianchi conjugate system to (1a). [Reviewer's remark: for another geometrical treatment of the Laplace system (1a) for two independent variables see G. Bol, Projektive Differentialgeometrie. I, Vandenhoeck and Ruprecht, Göttingen, 1950, pp. 120–125; these Rev. **11**, 539.] *V. Hlavatý* (Bloomington, Ind.).

Bochner, S. Euler-Poincaré characteristic for locally homogeneous and complex spaces. Ann. of Math. (2) **51**, 241–261 (1950).

This paper contains a number of theorems, observations and remarks on the characteristic of Riemannian manifolds based on the generalized Gauss-Bonnet formula. A Riemannian manifold is called homometric (or locally homogeneous) if for any two points the metric forms in the neighborhood of these points become identical upon introduction of suitable coordinates. Two homometric spaces are antimetric if, in suitable coordinates, the g_{ij} have the same value and the same first derivatives, but have their second derivatives opposite in sign (all this at some point in each space). This implies that the Riemann tensors have opposite signs, and by way of the Gauss-Bonnet formula one finds the relation $\chi(r_1)=(-1)^k\chi(r_2)$ where $2k$ is the common dimension of the two spaces. A "geometric ensemble" consists of a simply connected compact manifold V with a transitive group of motions, and a simply connected non-compact manifold $\bar{S}$, also with a transitive group of motions, such that V and $\bar{S}$ are antimetric; in addition there are the spaces S subordinate to $\bar{S}$, i.e., those of which $\bar{S}$ is the universal covering; they are homometric, but usually not homogeneous. These ideas describe a "typical" situation (elliptic vs. hyperbolic), which occurred, e.g., in Cartan's theory of symmetric spaces; other instances are considered. Since it is known that $\chi(V)\geqq 0$, it follows that $(-1)^k\chi(S)\geqq 0$. The Ricci curvature, reasons for $\chi(V)=0$ and imbedding questions are briefly touched upon. In the second part complex manifolds with a Hermite-Kähler metric are taken up. The case where the metric is obtained locally from an immersion into a flat complex Euclidean space is studied; it is shown that in this case the integrand K of the Gauss-Bonnet formula satisfies $(-1)^kK\geqq 0$. This implies the main theorem, which states, briefly: if on a complex manifold S_{2k} there exist m $(\geqq k)$ vector fields, everywhere of maximum rank, satisfying the condition $\partial\zeta_\alpha^{(\sigma)}/\partial z_\beta=\partial\zeta_\beta^{(\sigma)}/\partial z_\alpha$ then $(-1)^k\chi(S_{2k})\geqq 0$, plus a statement on the meaning of $\chi=0$. For the proof one notices that the Abelian integrals associated with the vector fields can be used to imbed the universal covering space $\bar{S}$, locally one-to-one, into a complex Euclidean space.

The relation $(-1)^k\chi(S)\geqq 0$ is established in still another case: D is a domain in complex Euclidean space, G a transitive group of complex automorphisms of D. A Hermite-Kähler metric for D is constructed which is invariant under G. Now let Γ be a discontinuous subgroup of G such that $D/\Gamma=S$ is compact; it is shown again that the integrand K of the Gauss-Bonnet formula satisfies $(-1)^kK\geqq 0$.

An appendix discusses almost automorphic vectors; the situation is as in the main theorem, except that the vector fields on $\bar{S}$ are not invariant under the covering transformations, but almost automorphic in a sense generalizing the concept of almost periodic function; again the conclusion $(-1)^k\chi(S)\geqq 0$ is reached. The paper contains many other remarks and questions. *H. Samelson.*

NUMERICAL AND GRAPHICAL METHODS

Riegels, Fritz. Formeln und Tabellen für ein in der räumlichen Potentialtheorie auftretendes elliptisches Integral. Arch. Math. **2**, 117–125 (1950).

This paper is concerned with the functions

$$G_n(k^2)=(-1)^n\int_0^{\frac{1}{2}\pi}(1-k^2\sin\theta)^{-\frac{1}{2}}\cos 2n\theta\,d\theta.$$

Expansions are given in the forms

$$\text{(i)}\qquad G_n(k^2)=(1-k^2)^{-\frac{1}{2}}k^{-2n}\{f_nE(k^2)-g_nK(k^2)\}$$

with coefficients tabulated for expansions of f_n and g_n in powers of $k'^2=1-k^2$ up to those in k'^{14} and $n=7$;

$$\text{(ii)}\qquad G_n=\tfrac{1}{2}\pi\frac{(2n+1)!}{2^{4n}n!n!}\sum_{\nu=0}^{\infty}{}'\prod_{s}{}'\frac{4\nu^2-1}{4\nu^2-4n^2}k^{2\nu}$$

with 5 decimal coefficients to $\nu=15$, $n=7$ [the factor $(2n+1)!/2^{4n}n!n!$ is omitted by the author in his formula (11)];

$$\text{(iii)}\qquad k'^2G_n=1+c_{n2}k'^2+c_{n4}k'^4+\cdots+(\ln(4/k'))(d_{n2}k'^2+d_{n4}k'^4+\cdots)$$

with 5 to 6 figure coefficients up to $c_{n,19}$, $d_{n,19}$ for n up to 7. The tables give 4 decimal values of $k^{n/2}G_n$ for $n=0(1)3$, $k^2=0(0.01)0.9(0.001)0.999$.

A few errors have been noted by the reviewer. In (11) on p. 119 a factor $(2n+1)!/2^{4n}n!n!$ is needed before the $\sum$ to obtain agreement with (10). In the table that follows, the column for $n=0$ has errors running up to 35 units of the 5th decimal, and $n=1$ has 1 and 2 unit errors in this decimal. Other tables have not been examined fully, although a comparison of $k^{n/2}\cdot G_0$ with Jahnke and Emde's Tables of Functions, and Hayashi's Funfstellige Funktionentafeln, reveals discrepancies in the 4th decimal of a unit or so in several cases, in which Riegels agrees with neither of the others.

J. C. P. Miller (London).

*Salzer, Herbert E. **Table of Powers of Complex Numbers.** National Bureau of Standards, Applied Mathematics Series, No. 8. United States Government Printing Office, Washington, D. C., 1950. iv+44 pp. $.25.

The table gives exact values for the real and imaginary parts of $(x+iy)^n$ for $n=1(1)25$ and $x, y=0(1)10$; $y=0$, $x=2(1)9$.

R. P. Boas, Jr. (Evanston, Ill.).

De Nockere, Georgette. Tables numériques des polynomes de Legendre $P_{n,0}(\cos\theta)$ et des fonctions associées $P_{n,j}(\cos\theta)$ ainsi que de leurs intégrales $\boxed{P}$ jusqu'à $n=15$ et $j=4$, pour l'argument θ (colatitude) variant de degré en degré. Tableaux des latitudes et longitudes divisionnaires et valeurs des multiplicateurs pour le calcul des coefficients du développement en série de polynomes de Laplace par la méthode des compartiments équivalents, d'une fonction de deux variables indépendantes. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **24**, no. 4, 166 pp. (1949).

The tables in this booklet give 5 decimal values of the Legendre polynomials and associated polynomials $P_{n,j}(\cos\theta)$, and of their integrals

$$\int_{t=\frac{1}{2}\pi}^{\theta} P_{n,j}(\cos t)\,d(\cos t)$$

for $n=0(1)15$, $j=0(1)\min(4,n)$ and $\theta=0(1°)90°$. First differences are given, although these are not, on their own, sufficient for interpolation to 5 decimals. The values of $P_n(\cos\theta)\equiv P_{n,0}(\cos\theta)$, $n=0(1)15$, and of $P_{n,j}(\cos\theta)$, $n=0(1)8$, have been reproduced from Tallqvist's tables [Acta Soc. Sci. Fennicae (1) **33**, no. 9 (1908); Nova Ser. A. **2**, no. 11 (1938); the latter reference, though mentioned, is not given in full], and these tables extended, largely by use of recurrence relations. The accuracy is erratic. Though they have not been examined, there seems no obvious reason to doubt that the values taken from Tallqvist are correct, and a partial search has revealed the following errors in the other tables. The gross errors seem confined to two tables and are at once apparent from the run of the differences given. In $P_{3,3}$, p. 54, the last two digits are entirely meaningless: the value at $\theta=0$ should be 8.83573, for instance. The table of $P_{4,4}$, p. 62, is also clearly wrong in its final two digits. The table of $P_{2,2}$, p. 48, is incorrect in its final digit, though trivially. It is clear that the tables should not be used without some kind of check.

J. C. P. Miller (London).

Katterbach, Klaus, und Krause, Helmut. Die Logarithmen der Integralexponentialfunktionen. Z. Astrophys. **26**, 137–146 (1949).

The tables give 5 decimal values of $\log_{10}K_n(x)$ for $n=1(1)5$, $x=0(0.01)1(0.02)2(0.1)6(0.2)10$, where

$$K_1(x)=-\mathrm{Ei}(-x)=\int_x^\infty t^{-1}e^{-t}dt,$$

$$K_{n+1}(x)=\int_x^\infty K_n(t)dt, \qquad n=1,2,3,\cdots.$$

First differences are given; these suffice for interpolation to full accuracy almost everywhere in the table. The tables are based on those given by G. Placzek [National Research Council of Canada, Division of Atomic Energy, Document no. MT-1 (1946); these Rev. **9**, 159]. *J. C. P. Miller.*

Amato, Vittorio. Su un procedimento di calcolo per l'applicazione dei metodi interpolatori e matrici di Vandermonde. Statistica, Milano **9**, 205–217 (1949).

In order to fit a polynomial $y=a_0+a_1x+\cdots+a_kx^k$ to n points (x_i, y_i) with $n>k+1$ the author constructs $k+1$ polynomials $P_j(\lambda)$, where $\lambda_1, \cdots, \lambda_n$ are n given real numbers, such that $\sum_{i=1}^n x_i^s P_j(\lambda_i)=0$ if $s<j$. Then the a_j are determined by the equations

$$\begin{aligned} a_0\textstyle\sum P_0(\lambda)+a_1\sum xP_0(\lambda)+\cdots+a_k\sum x^kP_0(\lambda)&=\sum yP_0(\lambda),\\ a_1\textstyle\sum xP_1(\lambda)+\cdots+a_k\sum x^kP_1(\lambda)&=\sum yP_1(\lambda),\\ \cdots\qquad\qquad &\cdots\\ a_k\textstyle\sum x^kP_k(\lambda)&=\sum yP_k(\lambda),\end{aligned}$$

which are readily solved. The method is worked out in detail for the case in which the principle of least squares forms the basis for the fitting process, and several numerical examples are worked. *W. E. Milne* (Los Angeles, Calif.).

Parkes, E. W. Linear simultaneous equations. Some practical aspects of their solution in respect to the time involved with a series and the relative accuracy of the results. Aircraft Engrg. **22**, 48, 56 (1950).

The author investigates the time T_n required to solve n linear simultaneous equations in n unknowns and concludes that, for sufficiently large values of n, $T_n\approx yn^4$, where y is a constant of the order of 0.001 hours for electrical calculating machines. He also examines the attainable accuracy and concludes that there is a loss of $0.3n$ significant figures between the initial coefficients and the values that are finally found for the unknowns. *W. E. Milne.*

Vinograde, Bernard. Note on the escalator method. Proc. Amer. Math. Soc. **1**, 162–164 (1950).

L'auteur donne un exposé de la méthode de l'escalier de J. Morris [The Escalator Method in Engineering Vibration Problems, John Wiley & Sons, Inc., New York, N. Y., 1947; ces Rev. **9**, 382] qui s'applique même si à la $p$ème étape, on rencontre des racines caractéristiques multiples. Quel que soit A_p, on peut toujours trouver une matrice inversable P_0 telle que $A_p=P_0A_0P_0^{-1}$ avec

$$A_0=\begin{bmatrix} \lambda_1 & \delta_1 & \theta & & & 0\\ 0 & \lambda_1 & \delta_2 & & & \\ & & \ddots & \ddots & & \\ & & & \lambda_1 & 0 & \\ & & & & \lambda_2 & \delta_{n_1} & 0\\ & & & & & \ddots & \ddots\\ 0 & & & & & & \end{bmatrix}.$$

Si

$$A_{p+1}=\begin{bmatrix} A_p & D \\ C & a_{p+1,p+1} \end{bmatrix}$$

on peut écrire $|A_{p+1}-\lambda I_{p+1}|=0$, I matrice unité, sous la forme

$$\begin{vmatrix} A_0-\lambda I_p & P_0^{-1}D \\ CP_0 & a_{p+1,p+1}-\lambda \end{vmatrix}=0$$

qui se réduit à l'équation donnée par Morris pour les racines caractéristiques de A_{p+1} lorsque A_p a toutes ses racines caractéristiques distinctes. L'auteur explique comment on détermine ensuite les vecteurs propres de A_{p+1}. A noter que J. Morris considère également des équations plus générales de la forme $|A-\lambda B|=0$. La note se termine par une condition nécessaire et suffisante pour que λ racine caractéristique de A_p soit racine caractéristique de A_{p+1}.

J. Kuntzmann (Grenoble).

Todd, John. The condition of certain matrices. I. Quart. J. Mech. Appl. Math. **2**, 469–472 (1949).

The author considers the problem of solving the equation $Z_{xx}+Z_{yy}=\lambda Z$ by difference techniques. He replaces this differential equation by the difference equations

$$Z_{r-1,s}+Z_{r,s-1}-4Z_{r,s}+Z_{r-1,s}+Z_{r,s+1}=h^2\lambda Z_{r,s},$$

where $Z_{r,s}=Z(rh, sh)$; here h is the length of the side of each elementary square in the mesh. The paper is devoted to a discussion of the "condition" of the matrix of coefficients of the difference equations; this is a measure of the precision with which the matrix can be inverted by numerical means. He uses two measures of "condition": (1) the so-called P-condition number, which is the ratio of the magnitudes of the largest to the smallest proper values of a symmetric matrix and (2) the so-called N-condition number, which is the norm of the matrix times the norm of its inverse divided by the order ν of the matrix. It is shown that the matrix of the difference system has P-condition number $O(\nu)$ and N-condition number $O(1)$. The corresponding results for certain second order total difference systems are $O(\nu^2)$ and $O(\nu^4)$. Thus, the solution of the difference system with two independent variables should be more stable than the corresponding system with one variable. In closing the author remarks that as the number m of independent variables increases the condition of the matrix improves and that this last result will be the subject of a future paper.

H. H. Goldstine (Princeton, N. J.).

Fricke, Arnold. Über die Fehlerabschätzung des Adamsschen Verfahrens zur Integration gewöhnlicher Differentialgleichungen 1. Ordnung. Z. Angew. Math. Mech. **29**, 165–178 (1949). (German. Russian summary)

The author deals with Adams's finite difference method for numerical solution of the first order equation (1) $y'=f(x, y)$ and improves the accuracy to which the maximum error can be predicted. Following Tollmien he uses the classical theorem of estimating the effect of an error $\vartheta(x, y)$ in $f(x, y)$ of equation (1) on the solution $y(x)$ by a term of the form $|\vartheta|_{\max} M^{-1}[\exp\{M|x-x_0|\}-1]$ where $|\partial f/\partial y|\leqq M$; a further error term $\delta\exp\{M|x-x_0|\}$ arises from an error δ in the prescribed initial value $y_0=y(x_0)$. Since Adams's approximation can be regarded as the exact solution of (1) with f replaced by its interpolation polynomial, $\vartheta(x, y)$ is estimated as an interpolation formula remainder term in both Adams "extrapolation" (advancing difference) method as well as in the "interpolation" (central difference iteration) method. An improved formula is obtained by gauging the maximum of $\int\vartheta dx$ rather than ϑ. The resulting gauges of accuracy are compared with the von Mises-Schulz error estimation and found to forecast smaller errors in many practical situations. The method does not allow for rounding off errors. The example $y'=(y-x)/(y+x)$ is treated in detail.

H. O. Hartley (London).

Stussi, F. Polygone funiculaire et équations différentielles. Bull. Soc. Roy. Sci. Liége **18**, 272–284 (1949).

The author develops a method of obtaining a numerical solution of the differential equation $y''+cy+F=0$ for a finite interval, where c is a constant and F is a given function of x. The interval is divided into n equal parts. The continuous function y is then replaced by a set of values y_m $(m=0, 1, 2, \cdots, n)$ obtained as follows. The function y is regarded as the load acting on a simply supported beam. The beam is then divided into n equal portions, each simply supported, and the load on each portion is assumed to vary linearly or quadratically. The values y_m are then defined to be the loads acting on each of the $n+1$ supports. The other functions in the above differential equation are treated similarly. By use of the theory of the funicular polygon, the set of values of y'' are expressed in terms of the set y_m. The above differential equation is then reduced to a set of algebraic linear equations for y_m. As an example, a solution is obtained when F is a constant, with $n=4$. In this case it is found that y_2 differs from its exact value by 0.13%. It is noted that when the same problem is solved by simple finite differences, y_2 has an error of 2.7%. The case when the differential equation contains y' can be treated similarly. It is stated that the method also applies when c is a given function of x.

G. E. Hay (Ann Arbor, Mich.).

Milne, W. E. A note on the numerical integration of differential equations. J. Research Nat. Bur. Standards **43**, 537–542 (1949).

The author offers a numerical method of integrating the ordinary differential equation $y^{(n)}(x)=f(x; y, y', \cdots, y^{(n-1)})$. The method is a step by step integration at equidistant intervals $x_{i+1}-x_i=h$. In the case $n=1$, with $y_i^{(m)}=y^{(m)}(ih)$ two consecutive steps are based on the formulae

$$(1)\quad y_1-y_0=\tfrac{1}{2}h(y_1'+y_0')-\tfrac{1}{10}h^2(y_1''-y_0'')+\tfrac{1}{120}h^3(y_1'''+y_0'''),$$

$$(2)\quad y_2-2y_1+y_0=7h(y_1'-y_0')-3h^2(y_1''+y_0'')+\tfrac{1}{12}h^3(11y_1'''-5y_0'''),$$

which are shown to be equivalent to 6th order Taylor formulae. Differentiating (1) (for $n=1$) yields

$$(3)\quad y''=f_x+f_yy';\quad y'''=f_{xx}+2f_{xy}y'+f_{yy}y'^2+f_yy''.$$

Based on (1), (2) and (3) a step by step process is described and illustrated with worked examples. The advantages claimed for the method consist of the simplification of formulae and elimination of the usual high order difference table. On the other hand, because of (3) the method is clearly limited to simple, and preferably linear, f. Generalisations to $n>1$ are given.

H. O. Hartley (London).

Collatz, L. Differenzenverfahren zur numerischen Integration von gewöhnlichen Differentialgleichungen n-ter Ordnung. Z. Angew. Math. Mech. **29**, 199–209 (1949). (German. Russian summary)

The author derives formulae for the numerical integration of the nth order differential equation

$$(1)\qquad y^{(n)}(x)=f(x; y, y', \cdots, y^{(n-1)}).$$

The method is a step by step integration based on a differ-

ence table of $y^{(n)}(x)=f$ at equidistant x_i (interval h). Having obtained values of the $y^{(m)}(x)$ for all $x_i \leqq x_r$, $m=0, 1, \cdots, n$, the $y^{(m)}(x_{r+1})$ are computed from

$$y^{(m)}(x_{r+1})=\sum_{\nu=0}^{n-m-1}(h^\nu/\nu!)y^{(m+\nu)}(x_r)+h^{n-m}\sum_{\rho=0}^{\rho}\beta_{n-m,\rho}\nabla^\rho f,$$

where the $\nabla^\rho f$ are backward differences of the f at $x=x_r$ and the β are tabulated numerical coefficients. The formula is therefore a Taylor expansion at $x=x_r$ in which all differentials up to $y^{(n-1)}$ are utilised (as they are in any case required for the evaluation of the f) and higher differentials are estimated from the difference table of the f. As an alternative to (1) a formula is offered in which the $\nabla^\rho f$ are backward differences at $x=x_{r+1}$ so that iteration is necessary but the tabulated new numerical coefficients β^* converge faster. In its present form the method is suitable when f involves all differentials but modifications are possible if some $y^{(m)}$ are not required. The link with other methods is established, the convergence of the iteration process investigated and the "starting off" of the solution briefly discussed. Note list of numerous errata. *H. O. Hartley.*

Schulte, A. M. A slight improvement of Southwell's method for the approximative computation of the lowest frequency of a homogeneous membrane. Appl. Sci. Research A. **2**, 93–96 (1950).

Southwell uses a numerical formula analogous to Rayleigh's quotient for the approximate computation of the lowest frequency. Schulte proposes a slight modification in the process and states that the new method reduces the computational work to about 80% of the work needed formerly. *W. E. Milne* (Los Angeles, Calif.).

Wassermann, G. D. A note on boundary perturbations. Proc. Cambridge Philos. Soc. **46**, 206–207 (1950).

The author modifies his treatment in an earlier paper [same Proc. **44**, 251–262 (1948); these Rev. **10**, 268], to take account of an error. The consequent derivation is considerably simplified. The formulae for the rth order change of the eigenvalue is given. *H. Feshbach.*

Kondo, Kazuo. On the practical solution of an integral equation with a singular kernel. Appl. Math. Mech. (Ōyō Sūgaku Rikigaku) **1**, 259–277 (1947). (Japanese)

The equation is

$$h(\eta)\varphi(\eta)+(2\pi)^{-1}\int_{-1}^{1}\frac{d\eta'}{\eta-\eta'}\frac{d\varphi}{d\eta'}=g(\eta).$$

According to a review in Appl. Mech. Rev., the author improves Multhopp's approximate method [Luftfahrtforschung **15**, 153–169 (1938)] by approximating the integral by a sum in which the division points are distributed more closely where the integrand is large.

Gray, Harry J., Merwin, Richard, and Brainerd, J. G. Solutions of the Mathieu equation. Trans. Amer. Inst. Elec. Engrs. **67**, 429–441 (1948).

The authors are concerned with solutions of the Mathieu equation in a less usual standard form, namely

$$d^2y/d\ell^2+\epsilon(1+k\cos t)y=0,$$

being interested in solutions when ϵ and k are given, rather than those having period 4π. Five decimal solutions $g(t)$, $h(t)$ are tabulated for $t=0(0.1)3.1$, π, with $\epsilon=1(1)10$, $k=0.1(0.1)1.0$. The solutions tabulated are such that $g(0)=1$, $g'(0)=0$, $h(0)=0$, $h'(0)=1$. Values are unrounded, and it is stated that the fifth decimal may be a little in error. No differences or derivatives are given. The values of the characteristic exponent μ, given by $\cos 2\pi\mu=g(\pi)h'(\pi)+g'(\pi)h(\pi)$, and of $M=\{-g(\pi)g'(\pi)/h(\pi)h'(\pi)\}^{\frac{1}{2}}$, each to 5 decimals, are collected in a separate table, which indicates well, by the many changes between real and complex values of μ and M, the need for further tabulation. *J. C. P. Miller.*

Malavard, Lucien, et Boscher, Jean. Sur la détermination numérique de fonctions biharmoniques par un procédé analogique. C. R. Acad. Sci. Paris **230**, 1493–1495 (1950).

On réalise au bassin électrique les deux fonctions harmoniques φ et ψ telle que la fonction biharmonique s'écrive $x\varphi+\psi$. Application à des problèmes d'élasticité plane. *Authors' summary.*

Herrera, Emilio. Flexi-calculateur pour intégrales et fonction elliptiques, son application au calcul de la "courbe de l'éclaireur." C. R. Acad. Sci. Paris **230**, 1134–1136 (1950).

This instrument takes advantage of the shape taken by a flexible strip to determine approximate values of the elliptic functions $E(\phi, k)$, $F(\phi, k)$, am (u, k), sn (u, k) and cn (u, k). One end of the strip is fixed in position and direction while the other end may be moved along a fixed track. It is applied to the determination of the curve of pursuit. *M. Goldberg* (Washington, D. C.).

Meyer-Eppler, Werner. Ein Abtastverfahren zur Darstellung von Ausgleichsvorgängen und nichtlinearen Verzerrungen. Arch. Elektr. Übertragung **2**, 1–14 (1948).

A device is suggested for evaluating the convolution integral

$$E(x)=\int_{-\infty}^{\infty}F(t)H(x-t)dt.$$

In it $H(q)$ is represented using Cartesian coordinates and a suitable scale by means of an opening in a plane. This plane is parallel to and between two other planes one of which contains a photographic plate and the other a rectangular source of light the intensity of which is proportional to $F(t)$ at the point whose abscissa is t. By this arrangement $E(x)$ is produced in the plane of the photographic plate without the necessity of any relative movement of the parts of the apparatus. With regard to this feature it may be contrasted with the quite similar device of R. Fürth and R. W. Pringle [Philos. Mag. (7) **37**, 1–13 (1946); these Rev. **8**, 287] and the cinema integraph of H. L. Hazen and G. S. Brown [J. Franklin Inst. **230**, 19–44, 183–205 (1940); these Rev. **2**, 62] neither of which are mentioned by the author. An illustration using the apparatus is shown but no physical details of its construction are given and no claim is made as to its accuracy. The projection on which it depends is used to investigate by geometrical means the product of certain time functions (arising in modulation) by considering the corresponding convolution in the frequency domain. *R. Church* (Annapolis, Md.).

Picone, Mauro. Nouveaux points de vue dans l'analyse des périodes. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 40–44. Centre National de la Recherche Scientifique, Paris, 1949.

The problem of finding periods for a function $f(t)$ defined in the interval $(-\infty, \infty)$ but actually known only in the

finite interval $(0, T)$ is associated with the problem of representing $f(t)$ in $(0, T)$ with a prescribed degree of accuracy by a linear combination of exponential functions $\exp(i\gamma_k t)$, $k=1, 2, \cdots, n$, where $n<N$, the value of N being fixed. This problem in turn is associated with the question whether or not $f(t)$ is the solution of a linear homogeneous differential equation with constant coefficients of order $n<N$. The author proposes two methods based on Gramian determinants for finding an answer to this question.

W. E. Milne (Los Angeles, Calif.).

Steffensen, J. F. On the technical functions of invalidity insurance. Skand. Aktuarietidskr. **32**, 160–175 (1949).

The computation of tables of invalidity insurance are as a rule rather laborious even if the rate of invalidity and the rate of mortality of the actives and the invalids are given as simple mathematical expressions. The author shows how the formulae become much simpler when you assume the ratio between the forces of mortality of the actives and of the mixed population of actives and invalids to be constant. After this paper was published it was found that the assumption led to very good results in an investigation of the statistics of the Danish social insurance. *P. Johansen.*

Zelenka, Ant. Rentrée en validité dans l'assurance-invalidité. Aktuárské Vědy **8**, no. 2, 76–87; **8**, no. 3, 95–114 (1949).

As a rule the formulae of invalidity insurance are worked out without involving the probability of reactivating, or rather, the decrement table of the invalids is chosen so that the actual value of the annuity of an invalid is reasonable. The author here shows how the formulae look when an explicit force of reactivity is introduced. Even though the forces of mortality and reactivity of the invalids are supposed to be aggregate (not depending on the time elapsed since the invalidity began) the formulae become rather heavy for practical use. *P. Johansen* (Lyngby).

Podtiaguine, N. Sur le calcul par groupes des réserves mathématiques dans l'assurance invalidité. Aktuárské Vědy **8**, no. 1, 7–21 (1948).

In case the rate of mortality and the rate of invalidity are graduated by the Makeham and Gompertz expressions, respectively, the Lidstone method of valuation can be generalized to the calculation of the values of invalidity insurance. *P. Johansen* (Lyngby).

MECHANICS

Bottema, O. Some theorems from kinematics. Simon Stevin **27**, 106–114 (1950). (Dutch)

By studying the motion of the plane at times t, $t+h$, $t+h+k$, the author obtains formulas which reduce to known results in plane kinematics when h and k are both made to approach zero. *W. J. Nemerever* (Stillwater, Okla.).

Meyer zur Capellen, W. Die Bahn des Momentanpols und die Kardanlage. Ing.-Arch. **17**, 308–316 (1949).

Mit einer Kardanlage eines ebenen kinematischen Systems ist eine Lage gemeint, welche momentan die einer Kardanbewegung ähnlich ist. Die unbestimmte Bedeutung des Wortes "ähnlich" hat zu Kontroversen über das Problem Veranlassung gegeben und die von Rauh, Marks, Bündgens, und Otto [Praktische Getriebetechnik, Band 2, Kardanbewegung und Koppelbewegung, VDI-Verlag, 1948] gegebenen Resultaten stimmen deshalb nicht mit den von Alt [Ing.-Arch. **14**, 319–331 (1944); diese Rev. **6**, 134] überein. Verf. bemerkt dass bei diesen Untersuchungen frühere Darstellungen von R. Müller und ihm selbst unberücksichtigt geblieben sind und betrachtet die Kardanlagen, besonders für das Gelenkviereck, von neuem. Er verlangt dabei dass in einer solchen Lage der Momentalpol eine Bahnstelle mit endlichem Krümmungshalbmesser beschreibt. Die Entartung der Mittelpunktkurve in den Wendekreis und die Polbahntangente bildet gewissermassen der Zentralpunkt des Problems und von diesem aus können die Zusammenhänge einfach und übersichtlich dargestellt werden. [Die Abhandlung zeigt einige Berührungspunkte mit der in derselben Zeit entstandene Arbeit des Ref., Nederl. Akad. Wetensch., Proc. **52**, 643–651 = Indagationes Math. **11**, 205–213 (1949); diese Rev. **11**, 217.] *O. Bottema* (Delft).

***Groenman, Jakob Tjakko. Behandeling van de koppelkromme met behulp van isotrope coördinaten. [Investigations on the Three-Bar Curve by Means of Circular Coordinates].** Thesis, Technische Hogeschool te Delft, 1950. 104 pp. (Dutch. English summary)

This is a monograph on the three-bar curve dealing in particular with the properties which have been found since 1920 by Hippisley, Bennett, Morley, Weiss, Alt, Mayer and others. The author has chosen the analytical treatment developed by Haarbleicher, who uses the isotropic lines through the origin as axes of the coordinate system. Much attention is given to the theorems on foci and nodes and to the fact that the curve is invariant for certain isogonal transformations. Special systems of three-bar curves are considered, such as the system of curves with common foci and nodes. Finally the theory is applied to some cases in which the foci and nodes have a particular position such as the curve with three cusps and that in which the foci and the nodes coincide. In this chapter the author gets some new results. *O. Bottema* (Delft).

***Beyerle, Konrad. Kreiselgeräte.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 211–233. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

A discussion of different kinds of gyroscopic equipment. The treatment is largely of an engineering rather than of a mathematical nature. *D. C. Lewis* (Baltimore, Md.).

Tzénoff, Iv. Quelques formes nouvelles des équations générales du mouvement des systèmes matériels. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 1, 13–16 (1949).

The reader may readily verify the identity,

$$-\frac{\partial T}{\partial x_i}=\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)-\frac{\partial \dot{T}}{\partial \dot{x}_i},$$

where T is a function of $x_1, \cdots, x_n, \dot{x}_1, \cdots, \dot{x}_n$ and

$$\dot{T}=\frac{dT}{dt}=\sum_{i=1}^{n}\left(\frac{\partial T}{\partial \dot{x}_i}\ddot{x}_i+\frac{\partial T}{\partial x_i}\dot{x}_i\right).$$

Hence Lagrange's equations may be written in the rather unfamiliar form

$$2\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)-\frac{\partial \dot{T}}{\partial \dot{x}_i}=Q_i.$$

This and similar identities are considered by the author,

leading to novel forms for the equations of motion of both holonomic and nonholonomic systems. *D. C. Lewis.*

Fogagnolo Massaglia, Bruna. Sulla stabilità di una configurazione di equilibrio di un sistema a due gradi di libertà. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **83**, 62–69 (1949).

The author undertakes to prove the following theorem. A configuration of equilibrium of a conservative dynamical system with two degrees of freedom is unstable if the potential energy is not a minimum at the configuration of equilibrium and if, of the four characteristic exponents, two are pure imaginary and two are zero. The proof involves the consideration of three subcases, and the treatment of one of these seems to the reviewer to require further elaboration. *L. A. MacColl* (New York, N. Y.).

Fogagnolo Massaglia, Bruna. Sugli esponenti caratteristici relativi ad una configurazione di equilibrio di un sistema a due gradi di libertà. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **83**, 70–81 (1949).

A configuration of equilibrium of a dynamical system with two degrees of freedom is stable if the four characteristic exponents have negative real parts. There is a well known set of conditions, due to Hurwitz, which is necessary and sufficient for the characteristic exponents to have this property. The author of the present note states that the Hurwitz conditions generally lead to rather complicated calculations, and she seeks a simpler set of conditions which shall be at least sufficient for the configuration of equilibrium to be stable. She finds three such sets of conditions, one of which is already known. The sets of conditions are of the same general character as the Hurwitz conditions, but they lend themselves more readily to physical application and interpretation. *L. A. MacColl.*

Cambi, Enzo. Sollecitazioni dinamiche di torsione. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **83**, 43–61 (1949).

The author considers a shaft which terminates at each end in a massive fly-wheel and, taking the elasticity of the shaft into account, he studies the motion of the system when a torque, which is an arbitrary function of time, is applied to one end of the shaft. It is assumed that the instantaneous configuration of the system can be adequately described by an angle α which is a function of the time and of the distance along the shaft. The problem is equivalent to a familiar problem concerning an electrical transmission line; and the solution is obtained in the usual way, by means of Fourier integrals and Laplace transforms. *L. A. MacColl.*

Hydrodynamics, Aerodynamics

***van den Dungen, F. H. Sur l'application du calcul des variations en mécanique des fluides.** Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 88–95. Centre National de la Recherche Scientifique, Paris, 1949.

This paper is a contribution to the classical problem of deriving the equations of motion of inviscid fluids from a variational principle, first partially solved by Clebsch, whose result was extended by Bateman [Proc. Roy. Soc. London. Ser. A. **125**, 598–618 (1929); cf. also Physical Rev. (2) **38**, 815–819 (1931)], the analysis being confined to the case of barotropic motion. Making use of two remarks of Bateman, the author obtains a variational principle valid in baroclinic motion; the associated invariant integral appears as the Bjerknes circulation theorem. Only the momentum equation is derived; the variational principle recently obtained by A. H. Taub [Proc. Symposia Appl. Math. **1**, 148–157 (1949); these Rev. **11**, 222] yields the energy equation as well, but is restricted to the case when heat conduction is neglected. The author discusses also the case of infinitesimal oscillations. In Bouligand's comments on the author's results reference is made to the corresponding problem for viscous fluids [C. B. Millikan, Philos. Mag. (7) **7**, 641–662 (1929)], recently reopened by H. Villat [Leçons sur les fluides visqueux, Gauthier-Villars, Paris, 1943, chapter III] and solved by R. Miche [J. Math. Pures Appl. (9) **28**, 151–179 (1949); these Rev. **11**, 62]. *C. Truesdell.*

***Painlevé, Paul. Leçons sur la résistance des fluides non visqueux. Rédigées par A. Métral et R. Mazet. Tome II, revu par R. Mazet.** Service de Documentation et d'Information Technique de l'Aéronautique, Paris, 1949. v+210 pp. 1000 francs.

[Volume I was published by Gauthier-Villars, Paris, 1930.] This volume is concerned with the movement of cylinders and three-dimensional bodies in an unlimited region of ideal incompressible fluid. The Joukowski hypothesis regarding the circulation about a cylinder is discussed in detail. Witoszynski's and other attempts to avoid certain philosophical difficulties of the Joukowski theory are discussed. The chapters on forces on three-dimensional bodies in accelerated motion follow classical lines. Theories of three-dimensional wings, such as Prandtl's, are not included. The book closes with a chapter, contributed by Métral, on the motion of a particle under the influence of gravity and a force proportional to speed squared. There are 28 pages of typographical errata appended to the book. *W. R. Sears* (Ithaca, N. Y.).

Oudart, Adalbert. L'étude des jets et la mécanique théorique des fluides. Publ. Sci. Tech. Ministère de l'Air, Paris, no. **234**, vii+177 pp. (1949).

The author gives an extensive exposition of the mathematical developments connected with the theory of jets. This includes the well known results for two-dimensional jets, developed on the basis of inviscid fluid theory (incompressible, subsonic, supersonic), and the turbulent jet. Related mathematical methods, such as the hodograph method and the method of characteristics, are considered in some detail. *C. C. Lin* (Cambridge, Mass.).

Truesdell, C. A. A new definition of a fluid. Naval Ordnance Laboratory Memorandum **9487**, Washington, D. C., 31 pp. (1948).

The purpose of the author is to establish the most general form of the stress tensor as a power series in the viscosity coefficient. First, through dimensional argument the author introduces three reference quantities, the reference viscosity, the reference temperature and a quantity which has the dimension of the gas constant in the equation of state for a perfect gas. The dissipation function is then considered to be a function of these three reference quantities and the mean pressure, the thermodynamic pressure, the temperature, the deformation tensor, the vorticity; the derivatives

of pressure, temperature, and external force field; and the molar concentrations of the components. The final form of the stress tensor is derived on the condition of isotropy and complete tensorial and dimensional invariance.

It seems to the reviewer that (1) the equations of motion used by the author are Eckart's equations which are probably incorrect [see Hirschfelder and Curtiss, J. of Chem. Phys. 17, 1076–1081 (1949)]; (2) the author's doubt on Burnett's equations derived from the kinetic theory is unfounded and arises from the neglect of pressure gradient in a pure rotation. *H. S. Tsien* (Pasadena, Calif.).

Truesdell, C. A. A new definition of a fluid. I. The Stokesian fluid. Naval Research Laboratory Report P-3457, Washington, D. C., iv+11 pp. (1949).

This is a more restrictive treatment than another paper by the author [see the preceding review]. Here a Stokesian compressible fluid is defined as a fluid whose dissipation function is only a function of the reference viscosity and reference temperature, mean pressure, thermodynamic pressure, temperature and deformation tensor.

H. S. Tsien (Pasadena, Calif.).

***Truesdell, C. A new definition of a fluid. I. The Stokesian fluid.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 351–364.

Aside from two remarks on the theory of elasticity and the Reiner-Rivlin theory of fluids with relaxation, this paper contains the same material as the author's two papers reviewed above. *H. S. Tsien* (Pasadena, Calif.).

Truesdell, C. A., and Prim, R. C. Zorawski's kinematic theorems. Naval Ordnance Laboratory Memorandum 9354, Washington, D. C., 4 pp. (1947).

This paper gives simple vectorial proofs of two theorems of Zorawski on the kinematics of vector tubes, answering the following questions. (i) Given a velocity field $\mathbf{v}(\mathbf{r}, t)$, what is a necessary and sufficient condition that the vector tubes of a second field $\mathbf{c}(\mathbf{r}, t)$ be material tubes? (ii) Given a velocity field $\mathbf{v}(\mathbf{r}, t)$, what is a necessary and sufficient condition that the strength of the vector tubes of a second field $\mathbf{c}(\mathbf{r}, t)$ remain constant throughout the motion?

C. C. Lin (Cambridge, Mass.).

Truesdell, C. A., and Prim, R. C. Vorticity and the thermodynamic state in the flow of an inviscid fluid. Naval Ordnance Laboratory Memorandum 9416, Washington, D. C., 14 pp. (1947).

This paper develops relations connecting the vorticity and the thermodynamic variables in the flow of inviscid fluids. Steady flow of a perfect gas is considered more in detail. Convenient forms of these relations are given from which simple conclusions can be drawn. *C. C. Lin.*

Truesdell, C. A. The kinematics of vorticity. Naval Ordnance Laboratory Memorandum 9591, Washington, D. C., 35 pp. (1948).

The following is an extract from the author's summary. Vortex motion is here presented simply as a branch of the kinematics of continuous media, without reference to any dynamical principles or to any special models. Part I is an exposition of those basic concepts and results in kinematics which are prerequisite to the analysis of vorticity. Part II introduces and interprets the vorticity field and quantities associated with it. Part III distinguishes between convection and diffusion of vorticity, discusses in detail the mechanism of convection, and closes with an exposition and critique of various proofs of Lagrange's velocity-potential theorem. *C. C. Lin* (Cambridge, Mass.).

Truesdell, C. A. On Ertel's theorem of the diffusion of vorticity. Naval Ordnance Laboratory Memorandum 9528, Washington, D. C., 8 pp. (1948).

The author's summary is as follows. A general diffusion equation involving the gradient of an arbitrary function is deduced. It is then specialized to the case of a viscous compressible fluid. In the case when viscosity may be neglected, it reduces to a theorem of Ertel [Naturwissenschaften 30, 543–544 (1942)]. From the Ertel theorem it is shown that the mixing of two inviscid incompressible fluids leads to a motion which is unstable, except in the special cases of plane and axially symmetric motion. This result may explain why the surface of separation for a spherical explosion in water is much smoother than that for a similar explosion in air. The optimum form for the estuary where a fresh water stream meets the ocean is discussed. The Ertel theorem is applied also to show that the specific vorticity generated at the head shock of a slightly yawed projectile increases as the air passes into the wake.

C. C. Lin (Cambridge, Mass.).

Kondo, Kazuo. On the vortex theory and the acceleration potential. J. Jap. Soc. Appl. Mech. 3, 10–14 (1950).

The object of this paper is to demonstrate the equivalence of the two approaches mentioned in the title. The paper contains some significant errors which suggest a confusion in the mind of the author as to the roles of the Eulerian (relative to a fixed point in space) and the Lagrangian (relative to a fluid particle) viewpoints in hydrodynamics. The velocity potential Φ and the acceleration potential ϕ are Eulerian and Lagrangian quantities, respectively, so $\partial\Phi/\partial t \neq \phi$. The author's equations (1.1), (1.2), and the right-hand formula at the bottom of p. 12 are therefore invalid as they stand. Much of the subsequent material can be salvaged, but the operation is much more elaborate than merely replacing $\partial/\partial t$ by D/Dt throughout, and may appropriately be done only by the author. *E. Pinney.*

Sears, W. R. Potential flow around a rotating cylindrical blade. J. Aeronaut. Sci. 17, 183–184 (1950).

An infinitely long cylinder, of arbitrary section, is considered to rotate uniformly, about an axis normal to it, in an incompressible inviscid fluid "otherwise at rest." The velocity field is determined simply in terms of the two-dimensional potential of the flow of a uniform stream normal to the cylinder. A special case is considered. Trailing vortices are absent. *M. J. Lighthill* (Manchester).

Trilling, Leon. The impact of a body on a water surface at an arbitrary angle. J. Appl. Phys. 21, 161–170 (1950).

The problem described in the title is treated with the following restrictions: viscosity, surface tension, compressibility and gravity are neglected; the free surface condition is linearized to become $\Phi = 0$ on the still water surface where Φ is the velocity potential (in particular, the effect of the splash is neglected); the body is assumed to move with constant velocity and no rotation. Because of the linear boundary conditions the potential may be expressed as a sum of two potentials arising from the horizontal and vertical components of the velocity of the body. The following shapes are treated exactly, where in each case the body is

started in motion impulsively when half submerged: infinitely long elliptic cylinder, sphere, ellipsoid of revolution, and general ellipsoid. The author discusses the application of these results to the problem of finding approximately the drag history as a body enters water up to the point when an entrance cavity begins to form. Application is made to the case of a sphere entering at a 45° angle. The general approach follows closely that of Sedoff [Trudy Central. Aerogidrodinam. Inst. no. 187 (1934)]; however, Sedoff worked out, as specific examples, only two-dimensional cases. *J. V. Wehausen* (Providence, R. I.).

Sretenskiĭ, L. N. The plane problem of the propagation of waves in a basin, excited by an underwater source. Izvestiya Akad. Nauk SSSR Otd. Tehn. Nauk **1950**, 321–332 (1950). (Russian)

The velocity potential and form of the free surface are derived for a line source of oscillating strength located on the bottom of a long rectangular basin. The problem is treated two-dimensionally and the usual linearized free surface condition is used. Approximate formulas for the free surface are derived for the case when the depth-to-breadth ratio is small. A continuous distribution of sources along an interval of the bottom is also considered and an approximation to the free surface derived. The methods for deriving the approximations follow closely those used in an earlier paper [Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **13**, 473–496 (1949); these Rev. **11**, 480].

J. V. Wehausen (Providence, R. I.).

Vedeler, Georg. A Mathieu equation for ships rolling among waves. I. Norske Vid. Selsk. Forh., Trondheim **22**, no. 25, 113–118 (1950).

Vedeler, Georg. A Mathieu equation for ships rolling among waves. II. Norske Vid. Selsk. Forh., Trondheim **22**, no. 26, 119–123 (1950).

With a more exact treatment of the hydrostatic forces than is customary the equation for the rolling motion of a ship in waves long compared with its beam leads to a differential equation of Mathieu's type. The author illustrates an approximate method of solution based in part on a paper of Brillouin [Quart. Appl. Math. **6**, 167–178 (1948); these Rev. **10**, 252], and for a special choice of parameters compares this approximate solution with two other possible ones. In deriving the equations of motion hydrodynamic forces are not considered and the ship is taken to be of infinite length. *J. V. Wehausen* (Providence, R. I.).

Supino, Giulio. Sul moto irrotazionale dei liquidi viscosi. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 615–620 (1949).

It was stated by Duhem [Recherches sur l'hydrodynamique, Gauthier-Villars, Paris, 1903, partie V, chapitre II, § 1] that an irrotational motion of a material which adheres to a rigid boundary is generally impossible, but his proof is not convincing. The reviewer [Naval Research Laboratory Rep. **3554** (1949); these Rev. **11**, 585] proved more generally that in a stationary finite domain to whose walls the material adheres everywhere without slipping, an isochoric motion in which the vorticity is zero or parallel to the velocity is impossible. The author obtains three theorems in part considerably stronger and in part somewhat weaker than those already known concerning this subject. He shows that an isochoric irrotational motion is impossible (1) if there is any finite stationary surface, however small, to which the material adheres, or (2) (subject to certain exceptions) if the material adheres to any part of a finite surface suffering rigid rotation or (3) for a viscous liquid whose only finite boundary is a single free surface. The proofs are based upon a theorem of Kirchhoff [Vorlesungen über mathematische Physik, v. 1, Mechanik, Teubner, Leipzig, 1876, Vorlesung 16, § 6] that if a function is harmonic in a region and vanishes along with its normal derivative upon any finite surface within the region, then it vanishes throughout the region. [Kirchhoff's proof is not rigorous, but can easily be amended.] *C. Truesdell* (Bloomington, Ind.).

Supino, Giulio. Sul moto irrotazionale dei liquidi viscosi. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 708–710 (1949).

[Cf. the preceding review.] The author shows that if in a motion of a viscous incompressible fluid the vorticity vector is irrotational, then the rate of dissipation of energy must equal the rate at which the viscous stresses do work (both in the interior of the fluid and on its boundaries). The objective of this note is apparently to answer an objection raised by Poincaré [Théorie des tourbillons, Paris, 1893, p. 194] against Navier's dynamical equation, founded upon the fact that in an irrotational motion of a viscous incompressible fluid the internal dissipation of energy vanishes. The objection has never seemed clear to the reviewer, nor does the author's explanation. *C. Truesdell.*

Viguier, Gabriel. Notions métriques liées à l'équation de Riccati et équation de Schrödinger intervenant dans l'étude du mouvement d'un fluide visqueux. C. R. Acad. Sci. Paris **230**, 1343–1344 (1950).

In the problem of one-dimensional unsteady flow of an incompressible viscous fluid under the influence of a body force, the equation of motion is shown to be reducible to either that of Riccatti or the special case of Schrödinger's equation for the motion of a linear harmonic oscillator.

Y. H. Kuo (Ithaca, N. Y.).

Štokman, V. B. Determination of the velocity of flow and distribution of density in the cross section of an infinite canal as they depend on the effect of the wind and the lateral friction in a Coriolis force field. Doklady Akad. Nauk SSSR (N.S.) **71**, 41–44 (1950). (Russian)

Casal, Pierre. Étude des champs de vecteurs aléatoires, appliquée à la cinématique des fluides turbulents. Bull. Soc. Math. France **77**, 141–147 (1949).

The author makes some basic but rather elementary remarks on homogeneous anisotropic turbulence from the point of view of the correlation tensor. *C. C. Lin.*

***Bass, J. Les bases d'une théorie statistique de la turbulence.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 212–226.

The author discusses the basic ideas of a statistical theory of turbulence. The change of spectrum F is discussed in the form $\partial F/\partial t = A_0 F$, where A_0 is an unspecified operator.

C. C. Lin (Cambridge, Mass.).

***Batchelor, G. K. Recent developments in turbulence research.** Proc. Seventh Internat. Congress Appl. Mech., 1948, introduction, pp. 27–56.

This is one of the general lectures at the Congress. The author chooses to limit himself mainly to a discussion of

the theory of decaying isotropic turbulence. Various developments of both theoretical and experimental nature are covered in a comprehensive manner. This includes Kolmogoroff's theory and its experimental confirmation, the spectrum and its change during the process of decay (mainly theoretical), and the change of turbulence intensity and other quantities during decay (theoretical and experimental). The final stage of the decay process, where our knowledge is almost complete, is discussed in detail, including both theoretical and experimental results. It is pointed out that it would be desirable to verify experimentally an asymptotic law of spectrum obtained according to Heisenberg's hypothesis. *C. C. Lin* (Cambridge, Mass.).

*Kampé de Fériet, J. **Le tenseur spectral de la turbulence homogène non isotrope dans un fluide incompressible.** Proc. Seventh Internat. Congress Appl. Mech., 1948, introduction, pp. 6–26.

This is one of the general lectures at the Congress. The author discusses the spectral tensors of energy and vorticity in homogeneous anisotropic turbulence and makes use of them to discuss the decay of turbulence, particularly in the case of no transfer of energy between various frequencies. The most general form of such a spectral tensor is obtained. *C. C. Lin* (Cambridge, Mass.).

*Burgers, J. M. **Problèmes se rattachant à la théorie de la turbulence.** Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 49–54. Centre National de la Recherche Scientifique, Paris, 1949.

An abbreviated version of the author's paper [Advances in Applied Mechanics, pp. 171–199, Academic Press, New York, 1948; these Rev. 10, 270]. *C. C. Lin.*

Burgers, J. M. The formation of vortex sheets in a simplified type of turbulent motion. Nederl. Akad. Wetensch., Proc. 53, 122–133 (1950).

The author considers a simplified model to demonstrate the formulation of sheets of concentrated vorticity in turbulent motion. A two-dimensional motion in the (y, z)-plane, described by the equations $\partial v/\partial t+v\partial v/\partial y+w\partial v/\partial z=0$, $\partial w/\partial t+v\partial w/\partial y+w\partial v/\partial z=0$ is made the basis of more elaborate discussions including, e.g., the viscosity. In this simplest form, domains with constant initial rates of deformation are considered. It is shown that some of them concentrate into lines, which are lines of sinks and at the same time vortex lines. Others eventually become symmetric divergent fields with the original vorticity infinitely diluted. A discussion is given to show the qualitative connection between this picture and the process occurring in turbulent flow. Further statistical investigations are proposed. *C. C. Lin* (Cambridge, Mass.).

Matsumoto, Toshizô. On Hayami's turbulent tensor. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. 24, 63–72 (1944).

Hayami considered the turbulent flow in the Yangtze River by using a modification of a formula of Gebelein [Turbulenz, physikalische Statistik und Hydrodynamik, Springer, Berlin, 1935] for the exchange coefficients, necessitated by the finite depth of the river. The author refines this modification and considers its consequences. *C. C. Lin* (Cambridge, Mass.).

Sears, W. R. The linear-perturbation theory for rotational flow. J. Math. Physics 28, 268–271 (1950).

The linear perturbation theory of the steady, plane or axisymmetrical, flow of a perfect gas, which varies only slightly from a uniform stream, and in which viscosity and heat transfer are neglected, but the entropy is not assumed uniform, is set up. However the theory is more complicated than the more general theory of the reviewer [Philos. Mag. (7) 40, 214–220 (1949); these Rev. 10, 641], since the author has not observed that the familiar theory (with uniform entropy) holds, with hardly any change, if therein the velocity component in the direction of the stream be replaced by a certain simple expression (involving the pressure) to which, on that theory, it is equal. *M. J. Lighthill.*

Tempest, R. K. The supersonic flow of compressible fluid through axially symmetric tubes of uniform and varying section. Proc. Roy. Soc. London. Ser. A. 200, 511–523 (1950).

Some solutions of the linearised equation of homentropic gas flow are used to deduce supersonic flows in axi-symmetrical tubes. However, the paper is unreliable. Thus in § 3 it is "proved" that no irrotational flow can exist if the cross-section has minimum area at entry and, over that cross-section, the entry velocities fall monotonically between the axis and the wall. This false conclusion is arrived at by ignoring a discontinuity of velocity gradient along a characteristic. *M. J. Lighthill* (Manchester).

Crocco, Luigi. Una nuova funzione potenziale per lo studio del moto bidimensionale non isentropico dei gas. Aerotecnica 29, 347–355 (1949).

This paper deals with the general problem of steady two-dimensional adiabatic flow of a gas in which, owing to the presence of shocks, the entropy is nonuniform. The author points out, following Reynolds, that the three quantities $p+\rho u^2$, ρuv, $p+\rho v^2$, where p is the pressure, ρ the density, and (u, v) the velocity, satisfy the equations of equilibrium of a distribution of plane stress. He then introduces $-\phi$ as the Airy stress function of this stress distribution (so that the three quantities mentioned are $-\phi_{yy}$, ϕ_{xy}, $-\phi_{xx}$), and calls ϕ a new "potential" of this (nonirrotational) flow. From its definition, and from the Bernoulli equation, the four quantities p, ρ, u and v are all expressible algebraically in terms of ϕ_{xx}, ϕ_{xy} and ϕ_{yy}. The equation of continuity then gives a quasi-linear third order partial differential equation for ϕ, whose three systems of characteristics are of course the Mach lines and the streamlines.

As this equation is somewhat complicated, an alternative formulation of the equations of motion is given, using the stream function ψ as well as the new potential; in this formulation the equations consist simply of $\psi_x=\rho v$, $\psi_y=-\rho u$, with the right hand sides replaced in terms of ϕ. The shock conditions may be expressed very simply, by saying that ϕ, ϕ_x and ϕ_y are continuous across a shock. The same is true for a vortex sheet. Outside a finite solid cylinder ϕ, ϕ_x and ϕ_y may return to different values on a path encircling it once; the changes involved are simply related to the forces and moment on unit length of the cylinder. The paper ends with the treatment of various known gas flows by the new method. *M. J. Lighthill* (Manchester).

Lighthill, M. J. The shock strength in supersonic "conical fields." Philos. Mag. (7) 40, 1202–1223 (1949).

One of the methods of approximate solution of the differential equations of isentropic irrotational flow is to seek the

velocity potential or velocity components as power series in a small parameter ϵ defined by the thickness of the body or its incidence. For example, if the flow is due to a stream of speed U, the velocity potential ϕ may be written $\phi = Uz[1+f(r,\theta)] = Uz(1+f_1+f_2+\cdots)$ where the subscripts indicate the power of ϵ involved. When the flow field is conical, the successive equations for f_i are of the form

$$(1-r^2)(f_i)_{rr}+r^{-1}(f_i)_r+r^{-2}(f_i)_{\theta\theta}=F. \tag{1}$$

Here r denotes $(M^2-1)(x^2+y^2)^{1/2}/z$ and θ denotes $\tan^{-1}(y/x)$; x, y, z are Cartesian coordinates, the z-axis being the conical axis (direction of stream flow), and F involves $f_{i-1}, f_{i-2}, \cdots, f_1$, and derivatives thereof. Equation (1) has a singularity at the Mach cone from the origin, $r=1$; $(f_1)_{rr}$ is infinite there, and consequently $(f_2)_r$ and all the successive $(f_i)_r$ are infinite there. Thus the expansion diverges in this region and it is impossible to study the flow there, or to satisfy the correct upstream boundary conditions, beyond the first order.

The author presents here a method by which the singularity is avoided in each step of the successive-approximation process. He assumes a coordinate transformation $R=r-r_1(\theta)-r_2(\theta)-r_3(\theta)-\cdots$ where again the subscripts indicate powers of ϵ. He is then able to choose each $r_i(\theta)$ in succession so as to make the coefficient of $(f_{i-1})_{rr}$ in F vanish. This permits convergence of the expansion near $R=1$. This process is to be carried out twice; once for $R<1$ and once for $R>1$. But R (for given r,θ) has a different meaning in these two regions, and the regions actually overlap, so that the solutions can be joined. By this technique the author is able to satisfy the shock-wave equations in the overlapping region. For cases where there is no disturbance ahead of the nose Mach cone it is found that $r_1(\theta)$ is zero. The shock strength is $O(\epsilon^2)$ at most. There are simple formulae for the shock strength and position, and for $r_2(\theta)$ and other quantities, in terms of the first-order solution. Cases where there is disturbance outside the Mach cone are also treated. As detailed examples, the flows past a slender cone, a flat triangular wing, and a flat rectangular wing tip are considered. *W. R. Sears* (Ithaca, N. Y.).

Weyl, Hermann. Shock waves in arbitrary fluids. Comm. Pure Appl. Math. 2, 103–122 (1949).

This paper is divided into two parts. In the first, the conditions are laid down which must be satisfied by the equation of state of a fluid in order that the shock phenomenon, with its usual characteristics, may occur. In the second, a gas satisfying these conditions is postulated and the nature of the interior of the shock layer itself is examined. The conditions which the author finds sufficient for shock formation in part I are in part local, in part topological in character. The local ones assert that pressure compresses the gas, and that compression becomes more difficult as the volume decreases. The topological conditions are to the effect that the manifold of possible states Z must be homeomorphic to a convex region in the (p,τ)-plane (p pressure, τ specific volume). Then corresponding to any state $Z_0(p_0,\tau_0)$ there corresponds a curve $Z(s)$ with the property that each ray from Z_0 cuts it exactly once, and the points $Z(s)$ comprise the totality of points Z connected with Z_0 by the Hugoniot relations. Thus Z_0 and $Z(s)$ are the states before and after a shock, s representing some such parameter as shock strength. The usual shock properties are then established.

In part II the full equations of mass, momentum and energy are considered with heat conduction and viscous stresses included. Two methods of allowing the thermal and stress parameters to vanish are discussed, one leading to a shock, the other to a slipstream. Only the former is considered further. This amounts to multiplying the coordinate x and the parameters by ϵ, and then letting $\epsilon\to 0$. The resulting equations reduce to two nonlinear first order equations in p and τ, involving the arbitrary equation of state of the gas (assumed only to obey the earlier conditions). The state Z_0 at $x=-\infty$ and Z_1 at $x=+\infty$ correspond, in view of the indefinite expansion of the x scale, to conditions on opposite sides of the shock. The author shows that these are both singular points, and indeed the only singular points of the system of differential equations. Moreover he proves that the initial point Z_0 is an unstable node, the final point Z_1 a saddle point. He points out that this particular combination of types would lead one to expect unicity of solution. [In point of fact, it has been indicated to the reviewer by L. A. MacColl that uniqueness can be proved without further assumption. Moreover, existence is assured if one is willing to discard certain topological possibilities which correspond to highly improbable physical situations.]

D. P. Ling (Murray Hill, N. J.).

Elasticity, Plasticity

Mušelišvili, N. I. Nekotorye osnovnye zadači matematičeskoĭ teorii uprugosti. [Some Fundamental Problems of the Mathematical Theory of Elasticity]. 3rd ed. Izdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad, 1949. 636 pp.

The first and second editions appeared in 1933, 1935. The author states that the general plan of the book is the same, but that most of it has been rewritten; integrals of Cauchy type are more fully discussed (a special chapter is now devoted to them); there is a completely new chapter on the solution of boundary problems of the plane theory of elasticity by reduction to a problem with a linear relation between the boundary values on either side of the boundary (Hilbert's problem).

Gross, Wolf. Sulla matrice di Green di un problema di elasticità piana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 444–448 (1949).

The paper is devoted to the well-known problem of finding the stresses and displacements in the interior of a circular disc for the case of concentrated loads, balanced by suitable distributions of surface tractions, all boundary forces being in the plane of the disc. The author's procedure of constructing singular solutions of the equations of plane elasticity for regions of this type by exhibiting the so-called Green's matrix corresponding to the first boundary-value problem (stress prescribed at the boundary) of the circle in plane elasticity is apparently new.

Quoting only the work in Love [A Treatise on the Mathematical Theory of Elasticity, 4th ed., Cambridge University Press, 1927, pp. 209–211], with which he is apparently not satisfied, the author asserts that no rigorous discussion of the singularities of the displacement vector for the case of concentrated loads is available in the literature of plane elasticity. That this statement is inaccurate is shown by the following references, without discussing the merit of the examples in Love: I. S. Sokolnikoff, Mathematical Theory of Elasticity, Brown University Notes, Providence, R. I.,

1941, section 65 (general theory of concentrated forces and couples based on the work of N. I. Muskhelišvili); B. Sen, Proc. Roy. Soc. London. Ser. A. **187**, 87–101 (1946); these Rev. **8**, 241; H. Poritsky, Trans. Amer. Math. Soc. **59**, 248–279 (1946); these Rev. **7**, 449. *A. W. Saénz.*

Rivlin, R. S. Large elastic deformations of isotropic materials. VI. Further results in the theory of torsion, shear and flexure. Philos. Trans. Roy. Soc. London. Ser. A. **242**, 173–195 (1949).

This paper continues the application of the author's theory of large elastic deformations [same Trans. Ser. A. **240**, 459–490, 491–508, 509–525 (1948); **241**, 379–397 (1948); Proc. Roy. Soc. London. Ser. A. **195**, 463–473 (1949); Proc. Cambridge Philos. Soc. **45**, 485–487 (1949); these Rev. **10**, 168, 340, 650]. The bodies considered are incompressible cylinders and cuboids, and the formulae are first developed with a stored energy W which is a general function of the two strain invariants I_1, I_2, with later specialization to $W=C_1(I_1-3)+C_2(I_2-3)$, where C_1, C_2 are constants. The author notes the agreement of his basic equations with those given by A. Signorini [Ann. Mat. Pura Appl. (4) **22**, 33–143 (1943); these Rev. **8**, 240]. In the case of a tube the deformation considered consists of a succession of a uniform simple extension, a uniform lateral inflation, and a uniform simple torsion, including as a special case that in which the tube is turned inside out. In the case of the cuboid or thick sheet simultaneous simple flexure and uniform extension normal to the plane of flexure are considered and some of the author's previous results generalized. *J. L. Synge.*

Murnaghan, Francis D. A revision of the theory of elasticity. Anais Acad. Brasil. Ci. **21**, 329–336 (1949).

A brief account of the author's theory of elasticity for finite deformations [Proc. Symposia Appl. Math., Vol. I, pp. 158–174, Amer. Math. Soc., New York, 1949; these Rev. **11**, 67]. *J. L. Synge* (Dublin).

Ziemba, S. Fonction des tensions aux coordonnées sphériques dans le cas d'une symétrie axiale des déformations et des tensions. Arch. Méc. Appl., Gdańsk **1**, 311–338 (1949).

The known differential equation satisfied by the stress function for axially symmetric problems in spherical polar coordinates is derived. [There is a misprint in its form given in equation (7), where the last term should have a positive and not a negative sign.] The stresses and displacements are laboriously deduced from the corresponding expressions in cylindrical coordinates. The well-known case, originally solved by J. H. Michell, of an axial force applied at the vertex of a cone is worked out. No references to existing literature are given. *B. R. Seth* (Delhi).

Green, A. E., and Zerna, W. Theory of elasticity in general coordinates. Philos. Mag. (7) **41**, 313–336 (1950).

This paper is an exposition of the theory of finite strain, the theory of stress, and the stress-strain relations for finite elastic strain. Its purpose is to present this material in tensor notation and the authors contend that their methods and results contain many new features not to be found in L. Brillouin [Les tenseurs en mécanique et en élasticité, Masson, Paris, 1938, chapter X] and F. D. Murnaghan [Amer. J. Math. **59**, 235–260 (1937)]. Another such exposition has been given by A. Tonolo [Rend. Sem. Mat. Univ. Padova **14**, 43–117 (1943); these Rev. **8**, 356], who translates into tensor formalism the homographic presentation of A. Signorini [Ann. Mat. Pura Appl. (4) **22**, 33–143 (1943); these Rev. **8**, 240].

The literature of this subject is now deplorably repetitious. Most of the basic results were obtained long ago by Cauchy, Green, St. Venant, Kirchhoff, and Finger in seldom quoted papers. Which system of presentation one employs is largely a matter of taste. The reviewer finds the methods of the present authors unnecessarily elaborate. One example: § 6 at some length derives the equations of motion for continuous media, but the results (6.18)(6.19) might have been written down by inspection from the classical result of Cauchy in Cartesian coordinates, since Cauchy's derivation is not restricted by any special hypotheses about the strain and is expressly referred to the deformed body. *C. Truesdell* (Bloomington, Ind.).

Zerna, W. Beitrag zur allgemeinen Schalenbiegetheorie. Ing.-Arch. **17**, 149–164 (1949).

The author's objective is to obtain the equations of the classical Love theory of thin elastic shells in as general a form and by as perspicuous a derivation as possible. He uses tensorial methods and exploits the first, second, and third fundamental forms of the middle surface to obtain the space metric in a coordinate system based on an arbitrary Gaussian net on the middle surface and on the normals to that surface. He defines the usual ten stress and moment resultants, and obtains the equilibrium equations in terms of them by direct application of the principles of statics, rather than by the shorter method of integrating the three-dimensional equilibrium equations in the coordinate system selected. After making the usual assumptions that a curve originally normal to the middle surface is deformed into a curve normal to the deformed middle surface, and that the normal stress across an element parallel to the middle surface is everywhere negligible in the stress-strain relations, he obtains approximate strain components and approximate stress-strain relations. By putting these quantities back into the definitions of the resultants, he obtains expressions for these quantities in terms of the displacements of the middle surface and their derivatives. When he specializes these results to the case of a shell of revolution, he obtains the formulae derived by the reviewer [Trans. Amer. Math. Soc. **58**, 96–166 (1945), § 8; these Rev. **7**, 231] by essentially the same method. These expressions contain a number of terms ordinarily neglected. *C. Truesdell.*

* **Vlasov, V. Z. Obščaya teoriya oboloček i eë priloženiya v tehnike. [General Theory of Shells and Its Applications in Technology].** Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad, 1949. 784 pp.

Grioli, Giuseppe. Sulla deformazione di un involucro cilindrico forato uniformemente premuto. Pubbl. Ist. Appl. Calcolo no. **246**, 5 pp. (1949).

With a view to applications in the design of steam heaters the author discusses the elastostatics of a thin cylindrical shell, with perforations, loaded by a uniform normal pressure in the outward sense. The problem is reduced to one of the calculus of variations and solved by the aid of Bessel functions J_1. *P. F. Neményi* (Washington, D. C.).

Stein, Manuel, and Mayers, J. A small-deflection theory for curved sandwich plates. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2017**, 20 pp. (1950).

Equations are developed for deflections and buckling of shallow portions of circular cylindrical sandwich shells. The

results take account of the effect of transverse shearing stress deformation and extend earlier work by Libove and Batdorf [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 899 (1948); these Rev. 11, 486] and the reviewer [J. Aeronaut. Sci. 15, 435–440 (1948); same Tech. Notes, no. 1832 (1949); these Rev. 10, 273, 653]. *E. Reissner.*

Ohlig, R. Die eingespannte Rechteckplatte. Ing.-Arch. 17, 243–263 (1949).

The author advances another approach to the solution of the problem of the bending of a thin elastic rectangular plate, with all four edges built-in, by means of an arbitrarily distributed load. Starting with the readily developed solution for the case where two opposite edges are pinned, and the other two built-in, an additional complementary solution of the well-known type $\sum_m \{Y_m(y)e^{imx}+X_m(x)e^{imy}\}$ is added to the expression for the plate deflection. The boundary conditions along the four built-in edges are formulated, and each is then developed into a Fourier series in terms of a coordinate running along the edge. Relations which must be satisfied by the arbitrary constants in the $Y_m(y)$ and $X_m(x)$ are then forthcoming in the usual manner. By an approximation method attributed to S. Iguchi [Eine Lösung für die Berechnung der biegsamen rechteckigen Platten, Springer, Berlin, 1933, p. 42] explicit expressions for these constants are obtained. Convergence of the resulting solution is discussed and representative calculations are carried out for the square plate under uniform load, and under a centrally placed concentrated load. *M. Goland.*

Uflyand, Ya. S. Integral equation for the bending of a sectorial plate with clamped radii. Doklady Akad. Nauk SSSR (N.S.) 72, 251–254 (1950). (Russian)

The author considers the following case: a thin horizontal plate, thickness h, in the form of a circular sector, radius R, central angle 2γ, clamped on the radial edges, and loaded vertically by a distributed load q. The problem is to determine the deflections of the plate and the maximum moment on one of the clamped edges. The series solution of the differential equation governing the deflection of the plate leads to two integral equations which reduce to one if the load q is symmetrical with respect to the axis of symmetry of the plate. The author discusses only this last case but includes also the situation when the edge on the arc of the sector is clamped; this gives very simple boundary conditions. The method is as for a plate simply supported on the edges; hypothetical distributed loads are applied along the radial edges in such a way that the deflections and their first derivatives at these edges equal zero.

The resulting integral equation is solved approximately by substituting concentrated forces for the hypothetical distributed loads along the radial edges. At the end numerical values are tabulated for the deflection at the center of the plate and the maximum moment on the clamped arc for the following conditions of loading: (1) uniformly distributed load q over the whole area of the plate; (2) concentrated load P at the center of the plate; (3) uniformly distributed load p along the central arc of the plate, $r=R/2$. All three cases are evaluated for $\gamma=\pi/8$, $\pi/4$, $\pi/2$. The deflections and the moments are given in a dimensionless form explained in the paper. *T. Leser.*

Payne, L. E. Torsion of composite sections. Iowa State Coll. J. Sci. 23, 381–395 (1949).

The torsion problem is considered for beams consisting of two different isotropic materials. In the cross section let C_1 and C_2 be two closed curves bounding the portions composed of the two materials. Solutions are then obtained in the cases when C_1 and C_2 are (a) concentric circles, (b) concentric ellipses, (c) confocal ellipses, (d) eccentric circles, (e) rectangles with a common side. In all cases the solutions are obtained readily in terms of the sort of functions encountered in the case of beams which are completely isotropic. The torsion problem is also considered in the case when one portion of the beam is isotropic and the other is orthotropic. Solutions are obtained as above in the cases when C_1 and C_2 are (a) concentric circles, (b) confocal ellipses, (c) equal rectangles with a common side. *G. E. Hay.*

Grigolyuk, È. I. Approximate solution of a problem on the stability of a ring under torsion. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 99–101 (1950). (Russian)

The author considers the following case: a thin circular plate of radius b with concentric hole of radius a, thickness of the plate δ, and torsional rigidity D, fixed on the outer and inner boundaries and turned by a moment M applied to the boundaries in the horizontal plane. The problem is to find the critical value of this turning moment for a given ratio of the radii a/b. The exact solution of the differential equation governing the deformation of a ring is inconvenient for numerical computations. The author presents it but uses it only for comparison with the results obtained from the approximate solution which he calls the Galerkin solution. At the end he tabulates critical values of the moment in dimensionless form, $k=M/\pi D$, for 6 ratios a/b. Two of those values are compared with the ones obtained from the exact solution and they differ by 12 and 7.2 percent respectively. *T. Leser* (Lexington, Ky.).

Goss, R. N. Center of flexure of beams of triangular section. Iowa State Coll. J. Sci. 23, 375–379 (1949).

B. R. Seth has solved the flexure problem for a beam in the cases when (a) the cross section is a triangle and the beam is incompressible, (b) the cross section is a right-angled triangle and the load is parallel to the hypotenuse. In the former case the author of the present paper has determined the center of flexure of the beam. In the latter case, a line on which the center of flexure lies is determined. *G. E. Hay* (Ann Arbor, Mich.).

Tsien, H. S. A generalization of Alfrey's theorem for viscoelastic media. Quart. Appl. Math. 8, 104–106 (1950).

The mathematical theory of the response of a "linear," incompressible, viscoelastic medium to nonhomogeneous, time-dependent stresses is generalized to cover the response of a compressible viscoelastic medium to such stresses. Body forces are also considered. *T. Alfrey, Jr.*

Savin, G. N., and Parasyuk, O. S. On some elastic-plastic problems with linear hardening. Doklady Akad. Nauk SSSR (N.S.) 70, 585–588 (1950). (Russian)

Stress distributions are determined for the plane strain problem which simultaneously satisfy the elastic problem as well as the plastic deformation equations of Hencky with linear hardening. The usual substitution of variables of St. Venant's theory is used, except that the intensity of shear stress $k(x, y)$ is now a function to be determined from the solutions and the mean normal stress in the plane ω is a harmonic function. General solutions are obtained for $k(x, y)$, which reduce to well-known solutions of plastic-elastic boundary value problems for special values of the arbitrary constants. *H. I. Ansoff.*

Rahmatulin, H. A. Investigation of the laws of propagation of plane elastic-plastic waves in a medium with a variable elastic limit. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 65–74 (1950). (Russian)

Plastic deformation theory is used to discuss propagation of plane longitudinal waves in a medium in which the elastic limit $\sigma(x)$ varies as a function of the coordinate. It is shown that the law of propagation of a plastic wave depends only on $\sigma(x)$ and the stress-strain curve and is independent of the manner of loading. In general, three stressed regions will exist in such a medium: an elastic region, a region of plastic loading and a region of unloading. For the case of linear hardening general solutions of the wave equation are given for each region in media with monotone varying elastic limit. Composite solutions are discussed for the nonmonotone cases. As an example, the elastic limit is determined as a function of length in an originally homogeneous isotropic elastic rod which is dropped on a rigid plate.

H. I. Ansoff (Santa Monica, Calif.).

Sokolovskiĭ, V. V. Plane and axisymmetric equilibrium of a plastic mass between rigid walls. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 75–92 (1950). (Russian)

Stress-strain relations of the plastic deformation theory are used with the hardening law given by $S=KE^{\mu}$, where S and E are the stress and strain intensities, respectively, and K and μ are constants. Assuming that only radial displacement takes place within the wedge, the components of strain are expressed in terms of E and a variable ψ which is a function of the wedge angle θ only. Differential equations for ψ for both the plane and the axially symmetric wedge are derived. Boundary values for ψ are obtained for $\mu\neq 0$ under the assumption of adhesion at the walls, and for $\mu=0$ under the assumption of a constant friction force at the walls. Numerical solutions are given for $\mu=1/2$ and $1/3$ for the plane wedge and $\mu=0$, $1/3$ and $2/3$ for the axially symmetric wedge. A solution in closed form is given for $\mu=0$ for the plane wedge. *H. I. Ansoff.*

Duwez, P. E., Clark, D. S., and Bohnenblust, H. F. The behavior of long beams under impact loading. J. Appl. Mech. **17**, 27–34 (1950).

Plane cross-sections are assumed to remain plane, rotary inertia and shear deformation are neglected and the impact is supposed to impart a constant velocity to a single cross-section of an infinite beam. The plastic action predicted by ordinary beam theory is taken into account for two cases: ideal plasticity and linear strain hardening. However, local plastic deformation at the impact surface is neglected. It is found that in the plastic as in the elastic case y/t is a function of x^2/t where y is the beam deflection at a distance x from the point of impact at time t. Comparison is made between experiments on steel and copper beams and deflection predicted by theory. Agreement is good but the critical check of strain was not made. [See Dohrenwend, Drucker, and Moore, Proc. Soc. Exper. Stress Analysis 1, no. 2, 1–10 (1944).] *D. C. Drucker* (Providence, R. I.).

Ansoff, H. I. Forming of a plastic sheet between fixed cylindrical guides with Coulomb friction. Trans. A.S.M.E. **72**, 145–152 (1950).

The problem considered is that of the drawing of a plastic sheet between two fixed cylindrical surfaces under the assumption that the material between the cylinders is in a state of plane plastic flow and behaves according to the Saint Venant-Mises theory. The general equations of plane plastic flow are discussed and their reduction to a system of quasi-linear partial differential equations given. The boundary conditions, including that of Coulomb friction along the cylindrical surface, are given. The equations are solved by the method of characteristics. Three basic regions in the sheet must be considered in the solution. The first turns out to be a "region of constant stress," the second represents a "noncentered fan," and the third region requires a numerical solution. Two methods are given for this region: approximation by a system of linear difference equations, construction of the characteristic curves by using approximate radii of curvature. The latter method was suggested by W. Prager [Courant Anniversary Volume, pp. 289–300, Interscience, New York, 1948; these Rev. **10**, 82]. The results show good agreement with those obtained by previous writers. The author also shows that in the practically important cases of small sheet thickness to cylinder radius ratios, the solution can be obtained by a very simple approximation. The results, thus obtained, agree extremely well with those of Sachs [G. Sachs and L. J. Klinger, J. Appl. Mech. **14**, A-88–A-98 (1947)]. The paper concludes with a discussion of the determination of the elastic-plastic boundary. *G. H. Handelman* (Pittsburgh, Pa.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Weinstein, W. Wave-front aberrations of oblique pencils in a symmetrical optical system: refraction and transfer formulae. Proc. Phys. Soc. Sect. B. **62**, 726–740 (1949).

The author discusses the aberration theory for the environment of a principal ray in an optical system with rotational symmetry by considering the deviation of the wave from a spherical wave through the sagittal focus. He develops formulae for tracing the second and (third order) aberration coefficients along the ray from surface (refraction and transfer formulae) and specializes his calculations for different azimuths. [Reference should be made to the basic work of A. Gulstrand, especially Svenska Vetenskapsakademiens Handligar, Ny Följd **63**, no. 13 (1924).]

M. Herzberger (Rochester, N. Y.).

Love, E. R. The electrostatic field of two equal circular co-axial conducting disks. Quart. J. Mech. Appl. Math. **2**, 428–451 (1949).

The field of the two disks, of radius a, equally or oppositely charged to potential $\pm V_0$, is shown to be given by the potential function

$$(1)\qquad \frac{V_0}{\pi}\int_{-1}^{1}[\{\rho^2+(\zeta+it)^2\}^{-1}\pm\{\rho^2+(\zeta'+it)^2\}^{-1}]f(t)dt,$$

the signs $+$, $-$ corresponding to the cases of equal and of opposite charges, ζa and $\zeta' a$ being the distances of the point from the plates, with a suitable sign convention, and ρa the distance from the axis, and each square root having positive real part. Here $f(t)$ is determined by the integral equation

$$(2)\qquad f(x)\pm\frac{1}{\pi}\int_{-1}^{1}\frac{\kappa}{\kappa^2+(x-t)^2}f(t)dt=1,\qquad -1\leqq x\leqq 1,$$

where κa is the distance between the plates. This is deduced in the first instance from a result of J. W. Nicholson [Philos. Trans. Roy. Soc. London. Ser. A. **224**, 303–369 (1924)], who gave the potential as a series of spheroidal harmonics, the coefficients being determined by an infinite system of linear equations. The author then verifies independently of Nicholson's work that the expression (1) satisfies the conditions of the problem. He shows in addition that the solution of (2) is given by the Neumann series, and that the capacity of either disc in either case is $(a/\pi)\int_{-1}^{1} f(t)dt$.

The latter part of the paper is devoted to a critical examination of Nicholson's procedure to solve the infinite system of equations and to derive an explicit solution. It is held that the procedure contains errors of principle, that the explicit solution contains divergent integrals which have yet to be elucidated, and that the method is incapable of rectification. *F. V. Atkinson* (Ibadan).

Miliyančuk, V. S. On a possibility for the construction of a generalized linear electrodynamics. Doklady Akad. Nauk SSSR (N.S.) **71**, 871–874 (1950). (Russian)

The author puts forward some general remarks concerning generalizations of electrodynamics in which the field equations are linear. Such generalized theories are derived from Lagrangians of the form $L=-(1/16\pi)f_{\mu\nu}F_{\mu\nu}+(1/c)s_\mu\phi_\mu$, where $f_{\mu\nu}$ is the electromagnetic field-tensor and $F_{\mu\nu}$ the displacement tensor, the two being related by a linear equation of the form $f_{\mu\nu}=(1+a_1\Box+a_2\Box^2+\cdots)F_{\mu\nu}$. A particular case of such a theory is obtained if one considers $F_{\mu\nu}$ to be an external electromagnetic field and $f_{\mu\nu}$ the effective field including the external field plus all the vacuum-polarization effects to which the external field gives rise. Thus the proposed type of theory comprises some features of classical theories of extended charges, and other features belonging to quantum electrodynamics and positron theory. *F. J. Dyson* (Princeton, N. J.).

Bosquet, Jean. Introduction à l'électrodynamique des conducteurs en mouvement. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **21**, no. 4, 85 pp. (1948).

The author demonstrates that the extension of Maxwell's field equations to moving conductors is readily made in rigorous form if one uses the system of Lorentz forces on the electrons within the conductor and on the positive ions constituting the rigid framework of the conductor, differentiating carefully between the absolute velocity of the electrons and the relative velocity with respect to the conductor. The dynamic equation of the electron motion includes a reaction force which accounts for the interaction between ions and electrons. In order to arrive at a general form for Ohm's law, the force in the direction of the electron motion is assumed proportional to the relative electron velocity [the booklet was apparently written before 1941 and a supplementary note to the introduction states that the author subsequently discovered that a similar idea was first used by H. A. Lorentz, "Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern," E. J. Brill, Leiden, 1895].

With the use of conventional vector and scalar potential functions the author then deduces a general law of induction for moving conductors which is applicable even in the case where the conductor does not form a closed loop. For closed loops the generalized law of induction reduces to the usual integral form of Faraday's law with the total time derivative of the magnetic flux. The author claims that the derivation is the only logical union of Maxwell's field equations, which were only stated for media at rest, and the basic laws of mechanics. In a further chapter the Lagrangian equations for systems of moving circuits are deduced and then applied to various types of electrical machines, to the condenser microphone and to the telephone receiver, considered as general electro-mechanical systems. The final chapter gives the relativistic formulation of Ohm's law and the generalized law of induction. *E. Weber* (Brooklyn, N. Y.).

Luneberg, Rudolf K. Asymptotic development of steady state electromagnetic fields. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. EM-14. ii+70 pp. (1949).

The paper presents two forms of asymptotic expressions for the steady-state electromagnetic field of periodic sources. These asymptotic expressions are built up from what is called the pulse solution of Maxwell's equations. The spatial distribution of the sources in the pulse solution is identical to that of the periodic sources in the steady-state problem, but the exponential time factor is replaced by a unit function $U(t)$ which is zero when $t<0$ and unity when $t>0$. The first of the asymptotic expressions is a power series in the wavelength whose nth term is determined by the sum of the discontinuities in the $(n-1)$th time derivative of the pulse solution. The second asymptotic representation, which should replace the first in case the discontinuities in the time derivatives of the pulse solution become infinite, is obtained by using a series of complex integrals, depending upon the singularities of the integrand which in turn depends upon the discontinuities of the pulse solution. Evaluation of these integrals leads to a series in the wavelength. Since both asymptotic expressions call for a knowledge of the discontinuities in the pulse solution and its derivatives, the author assumes that the full pulse solution is known.

The general result of the paper makes possible a clarification of the relationship of geometrical optics to electromagnetic theory. The theory of wave fronts, as developed by the author [Mathematical Theory of Optics, Brown University, 1944; these Rev. **6**, 107] says that the geometrical-optics values of the electromagnetic vectors at any point are the values on the wave front of the field created by the source or by the wave front of the pulse solution arising from the source. Thus the first term in the asymptotic expressions is essentially that given by geometrical optics. The next terms of the asymptotic expressions represent a correction to the approximation furnished by geometrical optics, so that the theory presented by the author may prove useful in many problems for which the steady-state solution is either difficult or impossible to obtain exactly. *C. J. Bouwkamp* (Eindhoven).

Jacottet, Paul. Elektromagnetische Wirbelfelder mit Schraubengestalt. Arch. Elektrotechnik **39**, 8–26 (1948).

The author obtains the complete solution for the electromagnetic field produced by two diametrically opposite thin helical wires located concentrically within a metallic cylindrical shell of finite thickness. The field components are derived from scalar potentials in the nonmetallic spaces and from Hertz vectors within the metallic cylindrical shell. The solutions involve the modified Bessel functions, as well as Hankel functions of complex argument. For the latter, approximate series developments are given for small argument involving the hypergeometric functions of Pochhammer. For the modified Bessel functions of argument of

the same magnitude as the order, a modification of the saddle point integration first introduced by P. Debye is developed.

The author also computes the eddy current losses in the cylindrical shell by means of the Poynting vector. By introducing appropriate simplifications, the final result can be interpreted as an increase in eddy current losses caused by skewing of the conductor pair; several special cases are tabulated. *E. Weber* (Brooklyn, N. Y.).

Bremmer, H. The propagation of electromagnetic waves through a stratified medium and its W.K.B. approximation for oblique incidence. Physica 15, 593–608 (1949).

The problem considered is that of a plane wave impinging on a horizontally stratified dielectric medium whose dielectric constant μ^2 is a function of the vertical coordinate z. A similar problem has been treated recently by O. E. H. Rydbeck [Trans. Chalmers Univ. Tech. Gothenburg [Chalmers Tekniska Högskolas Handlingar], no. 74 (1948); these Rev. 10, 252, 856], but the present approach is somewhat more general. Mathematically, the simplest formulation of the problem makes use of a modified Hertzian vector,

$$I=(\mu\cos\tau)^{\frac{1}{2}}\Pi_z=-(\chi_1 e^{i\xi}+\chi_2 e^{-i\xi})\exp ik_0(y\sin\tau_a-ct),$$

where $\xi=k_0\int_0^z\mu\cos\tau dz$, and the angle τ is defined as a function of z by the relation $\mu\cos\tau=(\mu^2-\sin^2\tau_a)^{\frac{1}{2}}$. The functions χ_1 and χ_2 satisfy the first order equations

$$d\chi_1/dz=f(z)e^{-2i\xi}\chi_2=\epsilon\lambda_1(z)\chi_2,\quad f(z)=\frac{1}{2}\frac{d}{dz}\log(\mu^{-1}\cos\tau),$$
$$d\chi_2/dz=f(z)e^{2i\xi}\chi_1=\epsilon\lambda_2(z)\chi_1;$$

while I satisfies the second order equation

$$d^2I/d\xi^2+(1-\epsilon dR/d\xi-\epsilon^2R^2)I=0,\qquad \epsilon R=\frac{1}{2}\frac{d}{d\xi}\log(\mu^{-1}\cos\tau).$$

The nonvanishing components of the electromagnetic field are obtained from Π_z by differentiation.

When $f(z)$ is a slowly varying function of z the functions χ_1 and χ_2 may be expressed as ascending series of powers of the auxiliary constant ϵ,

$$\chi_{1,2}=\sum_{n=0}^{\infty}\epsilon^n\chi_{1,2}^{(n)},\quad \frac{d}{dz}\chi_{1,2}^{(n)}=\lambda_{1,2}(z)\chi_{2,1}^{(n-1)}.$$

If we assume constant values for the first approximation, $\chi_1^{(0)}=C_1$, $\chi_2^{(0)}=C_2$, we obtain the W.K.B. solution, which is equivalent to one "ascending" wave and one "descending" wave in the stratified medium. Then the second approximation may be interpreted as additional waves of each type due to reflection losses in the W.K.B. wave, and so on. Similar results may be obtained by introducing integral equations for χ_1 and χ_2 instead of the differential equations.

For special values of μ^2 it may be possible to solve the equation for I in closed form. Thus, as a simple example, the author discusses the solution for $\mu=z_0/z$, $z>z_0$; $\mu=1$, $z<z_0$, in detail. The problem of the wave from a Hertzian dipole impinging on a horizontal ionosphere is also treated, and the integrals involved interpreted by a saddle-point method. *M. C. Gray* (Murray Hill, N. J.).

Försterling, Karl. Die Ausbreitung elektromagnetischer Wellen in einem geschichteten Medium unter der Mitwirkung eines Magnetfeldes bei schiefer Inzidenz. Arch. Elektr. Übertragung 3, 115–120 (1949).

The author derives a set of differential equations for the electromagnetic field in the ionosphere in presence of the earth's magnetic field. It is assumed that the ionosphere may be represented as a plane stratified layer whose dielectric constant is a function of height only; and further that at $z=\pm\infty$ the field is approximately equivalent to the sum of four partial waves, two incident components corresponding to a source at $z=\mp\infty$, respectively, and the other two corresponding to reflections of these incident waves. In the interior of the layer these four waves still exist, but they are not propagated independently. By means of rather complicated rotations of the coordinate axes the electric field components are expressed in terms of certain "transverse" electric displacement densities, transverse being used to indicate that they are perpendicular to a wave-normal which, however, has complex direction cosines. In the second part of the paper the magnetic field is neglected, and the dielectric constant is assumed to vary linearly with height. Then a solution in the neighborhood of $\epsilon(z)=0$ is derived in terms of Bessel functions of order $\pm\frac{1}{3}$, combined to give the correct behavior at $z=\pm\infty$. *M. C. Gray.*

Quantum Mechanics

***de Broglie, Louis. La Mécanique Ondulatoire des Systèmes de Corpuscules.** 2d ed. Gauthier-Villars, Paris, 1950. vi+223 pp. 1650 francs.

[The 1st ed. appeared in 1939.] This is a text-book giving an exposition of nonrelativistic wave-mechanics, with special emphasis upon the treatment of general dynamical systems involving many particles. The first chapter deals with the Hamiltonian theory of classical dynamics, and develops the formal analogy between classical dynamics and classical wave-optics. In the second chapter the Schrödinger equation is introduced on the basis of this analogy, and the main features of wave-mechanics are deduced from it. Subsequent chapters describe the methods of wave-mechanics and their application to specific problems. The book ends with an elementary account of the quantum statistics of systems containing several identical particles. *F. J. Dyson.*

Nishijima, K. On the spur calculations in quantum mechanics. Progress Theoret. Physics 5, 155–157 (1950).

The spur of a Hermitian operator of the form $F(Q)G(P)$, where Q and P stand for a set of f conjugate coordinates and momenta with continuous eigenvalues q and p, is equal to the integral of $F(q)G(p)/h^f$ over the $2f$-dimensional phase-space. This relationship is applied to the statistical mechanical problem of the partition function for a system of interacting identical particles. The relevant operator here is the Boltzmann factor, which is expanded under the assumption of high temperature to bring it into the necessary product form. The result expresses the quantum mechanical partition function in terms of averages over the classical phase-space. The paper ends with a discussion of a quantum mechanics in which the usual operators play the role of wave functions and the inner product of two operators is represented by the spur of their Hermitian product. *R. Karplus* (Cambridge, Mass.).

Bartlett, M. S., and Moyal, J. E. The exact transition probabilities of quantum-mechanical oscillators calculated by the phase-space method. Proc. Cambridge Philos. Soc. 45, 545–553 (1949).

Recently J. E. Moyal has shown [same vol., 99–124 (1949); these Rev. 10, 582] that quantum mechanics can

be represented in terms of functions defined in phase space. Infinitesimal transformations in time are given by the quantum analogue of Liouville's theorem; finite transformations by an integral operator. This method is used to obtain exact expressions for the transition probabilities of a perturbed oscillator. The usual expression obtained by perturbation appears in first approximation. *L. Tisza.*

Tolhoek, H. A., and De Groot, S. R. A general theorem on the transition probabilities of a quantum mechanical system with spatial degeneracy. Physica 15, 833–842 (1949).

The Ornstein-Burgers sum rule [e.g., E. U. Condon and G. H. Shortley, The Theory of Atomic Spectra, Cambridge University Press, 1935, p. 238] usually proved for allowed dipole transitions, is proved in a general fashion from the rotational invariance of the Hamiltonian. *L. Tisza.*

Snyder, Hartland S. Quantum electrodynamics: The self-energy problem. Physical Rev. (2) 78, 98–103 (1950).

"In this paper it will be shown that the limiting processes by means of which the integrals of the Hamiltonian of the interacting electron-positron field with the electromagnetic field are defined may be carried out so that the self-energies of the electrons and photons are zero" [author's summary]. The method consists in (i) deriving a formula for each self-energy which is a sum of several divergent integrals not all of the same sign, (ii) regarding the sum of the divergent integrals as the limit, as all ranges of integration tend to infinity, of a sum of integrals over finite ranges, (iii) choosing the finite ranges of integration to be different for the different terms in each sum. *F. J. Dyson.*

Ward, J. C. An identity in quantum electrodynamics. Physical Rev. (2) 78, 182 (1950).

A simple formal proof is given of an identity $Z_1=Z_2$ which was conjectured in a paper of the reviewer [Physical Rev. (2) 75, 1736–1755 (1949); these Rev. 11, 145]. The identity is required in order to prove that charge-renormalization in electrodynamics is an intrinsic property of the electromagnetic field and has the same value for charges carried by different types of particle. *F. J. Dyson.*

Nilsson, S. Bertil. Interaction of electrons and an electromagnetic field treated by analytic continuation. Ark. Fys. 1, 369–423 (1949).

This paper gives a comprehensive and critical account of the α-method of eliminating divergencies in quantum mechanics, including its relation to the λ-limiting process. The α-method was developed by Gustafson [see Ark. Mat. Astr. Fys. **34A**, no. 2 (1946), which has references to previous papers; these Rev. 8, 427] by making use of M. Riesz's approach to the solution of hyperbolic differential equations [Acta Math. 81, 1–223 (1949); these Rev. 10, 713]. In this method, a function depending on the real parameter α is obtained which remains finite and approaches a solution of the equation under consideration when α approaches a fixed limit. Nilsson extends the method by allowing α to take on complex values and is thus able to eliminate the logarithmic divergencies in the self-energy of the electron, a result not previously obtained by the α-method. He also derives the formula due to Bethe [Physical Rev. (2) 72, 339–341 (1947)] for the electromagnetic energy level shift in hydrogen.

Nilsson shows that the α-method gives the same numerical results as the λ-limiting process whenever the latter converges but is more powerful in that it sometimes gives finite results when the λ-limiting process diverges. However, he remarks that it is a disadvantage of the α-method, as so far developed, that in order to give it a Hamiltonian formulation it is necessary to abandon relativistic invariance except for the limiting value of α. He concludes with a brief discussion of the possibility of overcoming this defect by combining the α-method with that of Schwinger, giving a method of eliminating divergencies which is both relativistically invariant and works only with finite quantities.

A. J. Coleman (Toronto, Ont.).

Molière, G. Laufende elektromagnetische Multipolwellen und eine neue Methode der Feld-Quantisierung. Ann. Physik (6) 6, 146–162 (1949).

In scalar field theory the factor $r^{-1}e^{ikr}$ is characteristic of an outgoing spherical wave and $r^{-1}e^{-ikr}$ of an incoming wave. The author asserts that the usual procedure of carrying over this identification to vector field theories is not justified and that either of these factors may correspond to an outgoing wave, depending on its state of polarization. Except for the final section, the whole paper is devoted to a study of the polarization of spherical and plane waves. To this end, the author introduces a complex field vector $\mathbf{F}=\mathbf{H}-i\mathbf{E}$, where $\mathbf{H}$ is the magnetic and $\mathbf{E}$ the electric field strength, in terms of which Maxwell's equations take the form

$$i\partial\mathbf{F}/\partial t=\operatorname{curl}\mathbf{F}-4\pi\mathbf{j};\quad i\operatorname{div}\mathbf{F}=4\pi\rho,$$

where $\mathbf{j}$ is the current vector. In terms of an operator corresponding to the electron current operator of Dirac's theory, the energy, spin and total angular momentum are defined. A spherical wave corresponding to a multipole of arbitrary order is decomposed into three components which are (i) right circularly, (ii) left circularly or (iii) longitudinally polarized with respect to the direction of propagation of the wave. In Molière's treatment this decomposition replaces the usual distinction between magnetic and electric multipole waves. He introduces a polarization operator $\mathbf{P}$ which has the eigenvalues $+1$, -1 or 0 corresponding to the states (i)–(iii) of polarization of the photon. A plane wave is then expanded in terms of multipole spherical waves thus making explicit the components which correspond to outgoing or incoming waves.

In the last section, by applying the above to quantum field theory, Molière succeeds in eliminating zero-point energy. He remarks that in the transition from the classical Poisson brackets to the quantum mechanical commutator there is an ambiguity of sign. He takes advantage of this to introduce into the commutation relations the operator $\mathbf{P}$. Thus instead of the usual commutation relations, which in his notation are $[\mathbf{F}^*(\mathbf{r}), \mathbf{F}(\mathbf{r}')]=-8\pi\omega_{\mathbf{R}}\delta(\mathbf{r}-\mathbf{r}')$, where $\omega_{\mathbf{R}}$ is a differential operator, Molière takes the relations $[\mathbf{F}^*(\mathbf{r}), \mathbf{F}(\mathbf{r}')]=-8\pi\mathbf{P}\omega_{\mathbf{R}}\delta(\mathbf{r}-\mathbf{r}')$. With this change the zero point energy for noninteracting bosons vanishes.

There is a striking analogy between this theory and that of Rosenfeld and Solomon [cf. W. Pauli, Handbuch der Physik, 2d ed., v. 24, Teil I, Springer, Berlin, 1933, pp. 83–272, in particular, pp. 255–256]. Molière's has the advantage of employing for $\mathbf{F}$ a rather natural combination of $\mathbf{E}$ and $\mathbf{H}$ to which one is almost forced in discussing the physically meaningful property of polarization, whereas Pauli points out that the $\mathbf{F}$ of Rosenfeld and Solomon is rather artificial. On the other hand, as Molière remarks, his theory has the disadvantage of introducing into the commutation relations, which are generally regarded as

valid in any field theory, a function peculiar to the electromagnetic field, but he hopes this difficulty can be overcome since he believes there is an operator analogous to **P** in all field theories. *A. J. Coleman* (Toronto, Ont.).

de Wet, J. S. A note on the relativistic invariance of quantized field theories. Proc. Cambridge Philos. Soc. **46**, 316–318 (1950).

This note presents a simplification and an extension of an earlier proof [de Wet, Proc. Roy. Soc. London. Ser. A. **195**, 365–376 (1948); these Rev. **10**, 418] of the relativistic invariance of quantized field theories derived from a higher order Lagrangian. The simplification comes from the use of one-parameter families of spacelike surfaces. Then the invariance under coordinate transformations which change families of space-like surfaces into one another becomes the same as the consistency of the commutation rules with the equations of motion. The consistency in the case of Bose-Einstein quantization is proved for Hamiltonian densities which are polynomials in the field variables and their conjugate momenta; consistency of Fermi-Dirac quantization requires a Hamiltonian of the usual more special form.

R. Karplus (Cambridge, Mass.).

de Wet, J. S. The interaction representation in the quantum theory of fields. Proc. Roy. Soc. London. Ser. A. **201**, 284–296 (1950).

In this paper the interaction representation is considered from the point of view of the unitary transformation which connects the Heisenberg representation with the interaction representation. The equation of motion of the transformation operator is obtained from the equations of motion of the dynamical variables in the two representations. The latter equations are given in a Hamiltonian form where the usual Hamiltonian density is replaced by the invariant $T_{\nu}{}^{\mu}N_{\mu}N^{\nu}$; $T_{\nu}{}^{\mu}$ is the canonical energy-momentum tensor and N_{μ} is the time-like normal to the space-like surface associated with the interaction representation. The development in time of the transformation operator is found to depend on the difference between the scalar Hamiltonians in the two representations, both being of course expressed as functions of the dynamical variables in one of the representations. This difference is the interaction Hamiltonian. The uniqueness of the solution and its application to interacting scalar-meson photon fields are discussed.

R. Karplus (Cambridge, Mass.).

Sawada, Katuro. Note on the self-energy and self-stress. I. Progress Theoret. Physics **5**, 117–132 (1950).

In this paper the radiative self-energies and self-stresses are calculated for the following three pairs of interacting fields: pair-field with scalar mesons, pair-field with pseudoscalar mesons, and pair-field with vector mesons. The ordinary (in contrast to covariant) perturbation theory is used, and the expressions are examined to the second order in the coupling constant. Divergences are removed by consistent regularization. From his results the author concludes that the mass renormalization is consistent with a vanishing self-stress only for the pair-field and for mass-less mesons. Mesons with nonvanishing mass, therefore, are not relativistic particles in the present theory. Further physical interpretation of this result and of an expression for the self-stress in terms of a surface integral in the momentum space of the virtual particle are in preparation.

R. Karplus (Cambridge, Mass.).

Thermodynamics, Statistical Mechanics

Jardetzky, W. Zur Frage der Axiomatik des zweiten Hauptsatzes der Thermodynamik. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **5**, 33–47 (1939).

The author sets up a system of nine axioms in terms of mechanical concepts for the usual thermodynamical quantities, and develops the second law as a consequence.

C. C. Torrance (Annapolis, Md.).

Einbinder, Harvey. Further deductions from the K-theorem. Physical Rev. (2) **76**, 410–412 (1949).

The author has shown earlier [same Rev. (2) **74**, 803–805 (1948); these Rev. **10**, 156] that the equation of state of every n-dimensional ideal gas is $PV=sE(V,T)$ where $s=2/n$ in the nonrelativistic case ($E\ll mc^2$) and $s=1/n$ in the completely relativistic case ($E\gg mc^2$). This relation implies that the energy and Gibbs function are of the form $E(V,T)=T\phi(z)$, $G(V,T)=Tm(z)$ with $z=TV^s$. The Gibbs-Helmholtz equation leads to the differential equation $s\phi'-\phi=m'$, where the derivation occurs with respect to z. The integration of this equation leads to the thermodynamic functions of ideal quantum gases as power series in z, whereby the usual laborious eliminations are avoided. The case of the degenerate Bose-Einstein gas requires special discussion.

L. Tisza (Cambridge, Mass.).

Predvoditelev, A. S. A possible classification of statistical systems. Vestnik Moskov. Univ. **1947**, no. 7, 23–42 (1947). (Russian)

A criterion for the stability of motion of statistical systems is first established. Certain stepwise continuous functions associated with stable statistical systems are introduced. Identities, which are generalizations of those due to Riemann, Hugoniot and Hadamard, are found between the above functions. Combination of these identities with the Hamilton-Jacobi equation leads to equations of the Schrödinger type and of the Fokker-Planck-Einstein type, which may be easily transformed one into the other. The transformations used in establishing this connection enable the author to define canonical distributions of real, imaginary, and complex moduli. This leads to a natural classification of statistical problems in which classical and quantum statistical systems are shown to represent two aspects of the same general principle. *G. M. Volkoff.*

Leibfried, G., und Kaempffer, F. Über ein einfaches Verfahren zur Berechnung von Zustandssummen. Z. Physik **124**, 441–449 (1948).

The calculation of the distribution function of an ideal Bose-Einstein gas by means of the grand partition sum involves two summations: (1) over all occupation numbers of a single energy state, (2) over all energy states. Inverting the usual order the authors carry out the second summation first. The results admit a formal interpretation as representing a mixture of Boltzmann particles. The calculation is carried out explicitly for a system of linear oscillators obeying Bose-Einstein statistics. *L. Tisza.*

Kohler, Max. Behandlung von Nichtgleichgewichtsvorgängen mit Hilfe eines Extremalprinzips. Z. Physik **124**, 772–789 (1948).

D. Enskog [Thesis, Uppsala, 1917] has shown that the fundamental kinetic equation of the kinetic theory of gases can be written in the form of a variational principle. The

correct distribution function in the stationary state is characterized by the maximum value of the rate of increase of entropy per unit volume and unit time. Here the distribution function has to satisfy certain auxiliary conditions. The author shows that the same quantity is a minimum if constancy of the heat current and viscous stress are chosen as the auxiliary conditions. The two extremum principles are equivalent. They can be used as a basis for a Ritz approximation procedure. According to the function sets chosen one obtains the Enskog or the Burnett [Proc. London Math. Soc. (2) **39**, 385–430 (1935)] equations. The method is applied to the quantum statistical problem of metal electrons. *L. Tisza* (Cambridge, Mass.).

Rodriguez, A. E. A general kinetic theory of liquids. VI. The equation of state. Proc. Roy. Soc. London. Ser. A. **196**, 73–92 (1949).

An equation of state based on an approximation for the radial distribution function is studied. This equation contains explicitly the roots of a certain transcendental equation. The main purpose of the paper is the investigation of the roots of this equation. The original transcendental equation is approximated by an algebraic equation which eventually leads to explicit approximate expressions for the roots. The resulting equation of state has different characteristics from those obtained in the usual Mayer theory of condensation. In particular, the divergence of the cluster series is not found to occur at the density of the saturated vapor. An attempt (using the Lennard-Jones potential) is made to apply the theory to liquid argon.

J. M. Luttinger (Princeton, N. J.).

De Boer, J. Development of probability densities in power series of the density. Physica **15**, 680–688 (1949).

The probability of configurations consisting of an arbitrary number of molecules of a nonideal gas has been obtained by using classical mechanics for the molecules [J. E. Mayer and E. Montroll, J. Chem. Phys. **9**, 2–16 (1941)]. The author extends this theory to quantum mechanics. The method is the Ursell-Mayer expansion in terms of cluster integrals [J. E. Mayer, J. Chem. Phys. **5**, 67–73 (1937)]. The configurational probabilities and the kinetic energy density are obtained as power series of the particle density. General expressions are obtained for the caloric and thermal equations of state. The linear terms in the density are evaluated and are shown to be identical with those obtained with the partition function method.

L. Tisza (Cambridge, Mass.).

De Boer, J. The caloric and thermal equation of state in quantum statistical mechanics. Physica **15**, 843–848 (1949).

There are two statistical methods for obtaining the pressure of a fluid (thermal equation of state): (1) by using the volume derivative of the partition sum (thermodynamic pressure), and (2) by using the virial theorem, i.e., averaging the intermolecular forces (kinetic pressure). Born and Green have suggested [Proc. Roy. Soc. London. Ser. A. **191**, 168–181 (1947); these Rev. **9**, 402; ibid. **194**, 244–258 (1948)] that the two pressures may be different if quantum effects are of importance. They tried to base on this distinction an explanation of the strange phenomena in liquid helium II. The author claims that the two pressures are actually identical. The proof is based on the method of the paper reviewed above. Use is made of the rule that the trace of the commutator of a quantum mechanical operator vanishes.

L. Tisza (Cambridge, Mass.).

Green, H. S. The equation of state in quantized kinetic theory and quantum statistical mechanics. Physica **15**, 882–890 (1949).

Reply to de Boer's paper reviewed above. The author maintains his original claim about the difference between "thermodynamic" and "kinetic" pressure. He gives an evaluation of this difference and claims that de Boer's procedure for getting the trace of commutators equal to zero is not permissible for such singular operators as the momentum.

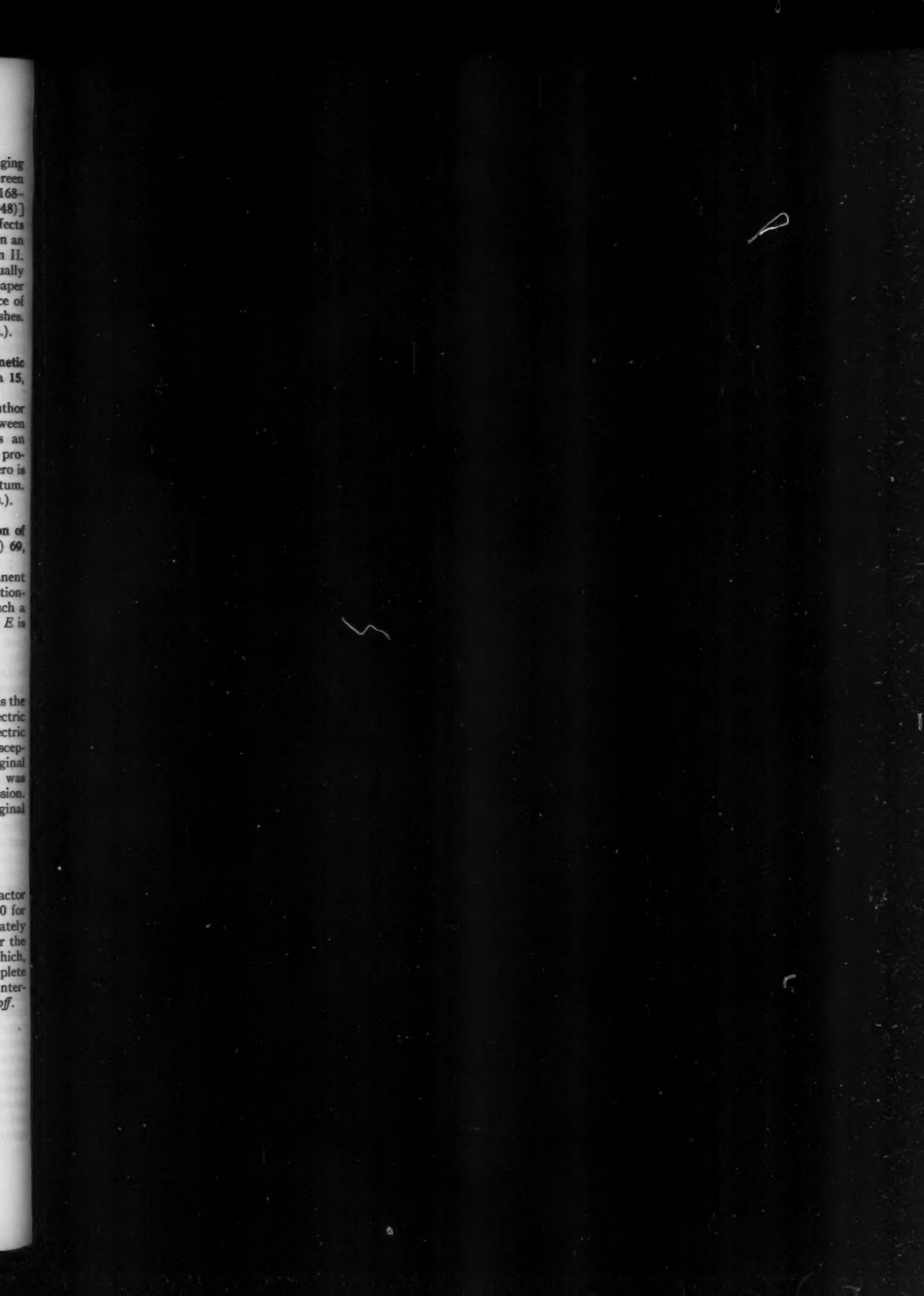
L. Tisza (Cambridge, Mass.).

Odelevskiĭ, V. I. On Onsager's theory of polarisation of dipolar liquids. Doklady Akad. Nauk SSSR (N.S.) **69**, 349–351 (1949). (Russian)

Molecules are considered which have both a permanent moment m_0 and an induced moment αE_e. The orientation-dependent term in the expression for the energy of such a molecule in a liquid placed in a constant external field E is shown to be

$$-U_\theta = m_1 G\cos\theta + \frac{4(\epsilon-1)}{2\epsilon+1}\,\frac{\epsilon(\epsilon_0-1)}{2\epsilon+\epsilon_0} m_1 E\cos\theta,$$

where $m_1=(\epsilon_0+2)(2\epsilon+1)/3(2\epsilon+\epsilon_0)m_0$, $G=3\epsilon E/(2\epsilon+1)$ is the field in a spherical cavity in the liquid, ϵ is the dielectric constant of the liquid, and ϵ_0 is the equivalent dielectric constant of a liquid whose molecules have the same susceptibility α, but no permanent moment m_0. Onsager's original work [J. Amer. Chem. Soc. **58**, 1486–1493 (1936)] was based on taking only the first term in the above expression. The inclusion of the second term modifies Onsager's original result for the dielectric constant:

$$\epsilon-\epsilon_0 \approx \frac{4\pi N_1 m_0^2}{kT}\,\frac{(\epsilon_0+2)^2}{6}$$

by multiplying the right hand side by an extra factor $(2\epsilon_0+1)/3$. The new expression gives approximately 50 for the dielectric constant of water instead of approximately 30 given by Onsager's formula. Although this is nearer the experimental result there is still a large discrepancy which, according to the author, is undoubtedly due to the complete neglect in both the above treatments of the local interactions of the permanent dipoles. *G. M. Volkoff.*